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Nonlinear analysis of inelastic frames considering a corotational approach and plasticity by layers: a discussion about computational implementation

Danilo B. Cavalcanti[1], Rafael L. Rangel[2], Luiz F. Martha[1]

[1]*Dept. of Civil and Environmental Engineering, Pontifical Catholic University of Rio de Janeiro*
Rua Marques de São Vicente 225, 22451-900 Gávea, Rio de Janeiro - RJ, Brazil
danilocavalcanti@aluno.puc-rio.br, lfm@tecgraf.puc-rio.br
[2]*International Centre for Numerical Methods in Engineering (CIMNE), Polytechnic University of Catalonia*
C/ Gran Capità s/n, 08034 Barcelona, Spain
rrangel@cimne.upc.edu

Abstract. This work presents the development of a numerical solver for nonlinear analyses of plane frame structures using the corotational formulation to take into account geometric nonlinearity and considering an elasto-plastic constitutive model. A one-dimensional rate-independent plastic model with isotropic hardening was implemented, and the cross-sections of beam-column elements were subdivided into layers to simulate plastification. The goal is to discuss the computational implementation aspects of the inelastic material behavior in an object-oriented framework for nonlinear finite element analysis of frame structural models. A numerical example is presented to highlight the influence of the inelastic behavior in the equilibrium path of structures.

Keywords: beam-column element, geometric nonlinearity, elasto-plasticity, object-oriented programming.

1 Introduction

The computational implementation of inelastic constitutive models, in the context of the Finite Element Method (FEM), is a vast topic and is addressed by numerous papers and books, such as de Souza Neto et al. [1], Borst et al. [2], Menétrey and Zimmermann [3], and Commend and Zimmermann [4]. However, few of these works discuss the implementation of plasticity for beam-column elements, here called frame elements. Therefore, the goal of the present work is to briefly describe and discuss the implementation of this type of material behavior in a structural analysis program dedicated to frame models. The NUMA-TF (*NUMerical Analysis of Trusses and Frames*) [5] program is open-source, developed in MATLAB, and based on the Object-Oriented Programming (OOP) paradigm.

The base solver for the implementations of this work already had formulations for geometrically nonlinear analysis with different kinematic descriptions of motion. The corotational formulation was chosen to couple the effects of geometric and material nonlinearity. However, although the program deals with planar and spatial structures, this work is limited to two-dimensional frames, made up by elements with prismatic cross-sections and flexural behavior described by the classical Euler-Bernoulli bending theory. It allows the adoption of a simple one-dimensional rate-independent plastic model with isotropic hardening. The integration of the constitutive equations is performed using a return-mapping algorithm. To simulate the plastic behavior along the cross-section of elements, a subdivision of the cross-sections into layers is made. The resulting numerical solver proved to be a modular application where new models can be easily incorporated and which can be easily coupled with other nonlinear effects.

The theoretical aspects on the coupling of geometric and material nonlinearities are presented in section 2. In section 3, the code structure is described with focus on the modifications made. Section 4 provides a numerical example of a Lee frame, which is used to validate the solver and highlight the influence of the inelastic behavior in the equilibrium path of structures. Finally, section 5 draws general conclusions and points towards future works.

2 Problem formulation

2.1 Corotational formulation

To consider the geometrically nonlinear behavior of elements, an appropriate kinematic description of motion must be adopted. In this work, the corotational approach is employed. It is based on two reference configurations to describe the motion of an element: the initial and the co-rotated configuration. The former is used to measure rigid-body displacements and rotations, while the latter is taken as a reference to measure the displacements and rotations that promote deformations and, consequently, internal forces to the element. Figure 1 illustrates this decomposition applied to a plane frame element. For more details, the interested reader is referred to Rangel [6].

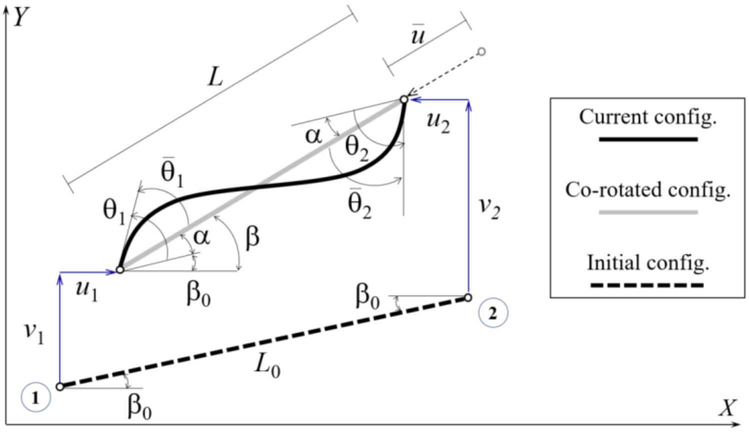

Figure 1. Kinematic description of motion of a plane frame element based on the corotational approach

The co-rotated configuration defines the so-called natural system of the element. The natural displacements and the corresponding natural forces are arranged in vectors, as presented in eq. (1), where N is the internal axial force, and M_1 and M_2 are the end-nodes bending moments. These vectors are related through the element tangent stiffness matrix in the natural system: $\mathbf{K}_n\,\mathbf{d}_n = \mathbf{f}_n$.

$$\mathbf{d}_n = \begin{bmatrix} \bar{u} & \bar{\theta}_1 & \bar{\theta}_2 \end{bmatrix}^{\mathrm{T}}, \quad \mathbf{f}_n = \begin{bmatrix} N & M_1 & M_2 \end{bmatrix}^{\mathrm{T}} \tag{1}$$

To solve the nonlinear problem, it is necessary to obtain the tangent stiffness matrix of each element in the global coordinate system, $\mathbf{K}_t$. The transformation of the tangent matrix from natural to global system is done as:

$$\mathbf{K}_t = \mathbf{B}^{\mathrm{T}}\,\mathbf{K}_n\,\mathbf{B} + \frac{N}{L}\mathbf{z}^{\mathrm{T}}\,\mathbf{z} + \frac{M_1 + M_2}{L^2}\left(\mathbf{z}\,\mathbf{r}^{\mathrm{T}} + \mathbf{r}\,\mathbf{z}^{\mathrm{T}}\right), \tag{2}$$

where matrix $\mathbf{B}$ relates the element nodal forces in the global system to the natural forces ($\mathbf{f}_g = \mathbf{B}^{\mathbf{T}}\,\mathbf{f}_n$):

$$\mathbf{B} = \begin{bmatrix} -\cos(\beta) & -\sin(\beta) & 0 & \cos(\beta) & \sin(\beta) & 0 \\ -\dfrac{\sin(\beta)}{L} & \dfrac{\cos(\beta)}{L} & 1 & \dfrac{\sin(\beta)}{L} & -\dfrac{\cos(\beta)}{L} & 0 \\ -\dfrac{\sin(\beta)}{L} & \dfrac{\cos(\beta)}{L} & 0 & \dfrac{\sin(\beta)}{L} & -\dfrac{\cos(\beta)}{L} & 1 \end{bmatrix}, \tag{3}$$

and vectors $\mathbf{z}$ and $\mathbf{r}$ are defined as:

$$\mathbf{z} = \begin{bmatrix} \sin(\beta) & -\cos(\beta) & 0 & -\sin(\beta) & \cos(\beta) & 0 \end{bmatrix}^{T}, \tag{4}$$

$$\mathbf{r} = \begin{bmatrix} \text{-}\cos(\beta) & -\sin(\beta) & 0 & \cos(\beta) & \sin(\beta) & 0 \end{bmatrix}^{T}. \tag{5}$$

The main reason for choosing the corotational approach is that it leads to a natural separation of geometric and material nonlinearity. In fact, geometric nonlinearity arises solely from the transformation between natural and global systems, and is taken into account by the second and third terms of eq. (2), the ones that depend on internal forces. These terms constitute the geometric stiffness matrix. The relationship between displacements and forces in the natural system, given by the natural tangent stiffness matrix, may assume small deformations, and it is where the material nonlinearity can be taken into account. As highlighted by Battini [7], with an appropriate element discretization, the strain level can be considered small without any loss of precision.

Note that none of the above equations assume any particularization for the local formulation of the element, allowing the consideration of different bending behaviors. In this work, Navier's beam theory is adopted. In this way, the only non-zero strain component is the axial strain, ε, computed as a function of natural displacements as in eq. (6), where x and y are, respectively, the axial and transversal coordinates along the element center-of-gravity:

$$\varepsilon = \frac{\bar{u}}{L} + y\left[\left(\frac{4}{L} - \frac{6}{L^2}x\right)\bar{\theta}_1 + \left(\frac{2}{L} - \frac{6}{L^2}x\right)\bar{\theta}_2\right]. \tag{6}$$

Applying the Principle of Virtual Work, the element natural forces are found to be the results of the following integrals of the normal stress, σ, acting on cross-sections:

$$N = \int_V \frac{\sigma}{L} dV, \quad M_1 = \int_V \left(\frac{4}{L} - \frac{6}{L^2}x\right)\sigma\, y\, dV, \quad M_2 = \int_V \left(\frac{2}{L} - \frac{6}{L^2}x\right)\sigma\, y\, dV\,. \tag{7}$$

The tangent stiffness matrix in the natural system is then computed as follows:

$$\mathbf{K}_n = \begin{bmatrix} \partial N/\partial\bar{u} & \partial N/\partial\bar{\theta}_1 & \partial N/\partial\bar{\theta}_2 \\ \partial M_1/\partial\bar{u} & \partial M_1/\partial\bar{\theta}_1 & \partial M_1/\partial\bar{\theta}_2 \\ \partial M_2/\partial\bar{u} & \partial M_2/\partial\bar{\theta}_1 & \partial M_2/\partial\bar{\theta}_2 \end{bmatrix}. \tag{8}$$

2.2 Elastic material model

When considering a material with linear-elastic behavior, the integrals of eq. (7) can be evaluated analytically. In this case, the natural tangent stiffness matrix is given by eq. (9), where E is the Young's modulus, A is the cross-section area, and I is the moment of inertia with respect to the global z-axis.

$$\mathbf{K}_n = \begin{bmatrix} EA/L_0 & 0 & 0 \\ 0 & 4EI/L_0 & 2EI/L_0 \\ 0 & 2EI/L_0 & 4EI/L_0 \end{bmatrix} \tag{9}$$

2.3 Elasto-plastic material model

When considering a material with elasto-plastic behavior, the integral expressions cannot be solved analytically, as the solution depends on the stress-strain history. Since small strains are assumed in the corotational approach, the axial strain can be additively decomposed into an elastic, ε^e, and a plastic, ε^p, part:

$$\varepsilon = \varepsilon^e + \varepsilon^p\,. \tag{10}$$

The elastic equilibrium is defined as follows, where the normal stress is taken as the Cauchy stress:

$$\sigma = E\left(\varepsilon - \varepsilon^p\right). \tag{11}$$

The yield function is given by a one-dimensional model with linear isotropic hardening behavior, as shown in eq. (12), where σ_Y^0 is the initial yield stress, E_p is the plastic modulus, and α is the isotropic hardening parameter.

$$f\left(\sigma,\alpha\right) = \left|\sigma\right| - \left(\sigma_Y^0 + E_p\,\alpha\right) \tag{12}$$

The evolution of the plastic strain follows the flow rule of eq. (13), where γ is the plastic multiplier, while the evolution of the hardening parameter follows the isotropic hardening rule of eq. (14).

$$\dot{\varepsilon}^p = \gamma \operatorname{sign}(\sigma) \tag{13}$$

$$\dot{\alpha} = \gamma \tag{14}$$

To integrate the constitutive equations, the elastic predictor-plastic corrector strategy is employed with the backward Euler integration scheme (sub-index n refers to current step):

$$\begin{cases} \sigma_{n+1} = \sigma_{n+1}^{\text{trial}} - E\,\Delta\gamma \operatorname{sign}(\sigma_{n+1}) \\ \varepsilon_{n+1}^p = \varepsilon_n^p + \Delta\gamma \operatorname{sign}(\sigma_{n+1}) \\ \alpha_{n+1} = \alpha_n + \Delta\gamma \end{cases} . \tag{15}$$

It starts by determining the trial elastic stress as:

$$\sigma_{n+1}^{\text{trial}} = \sigma_n + E\,\Delta\varepsilon_n . \tag{16}$$

The yield function is then evaluated with the trial elastic stress:

$$f_{n+1}^{\text{trial}} = \left|\sigma_{n+1}^{\text{trial}}\right| - \left(\sigma_Y^0 + E_p\,\alpha_n\right) \rightarrow \begin{cases} f_{n+1}^{\text{trial}} \le 0, & \text{elastic loading or unloading} \\ f_{n+1}^{\text{trial}} > 0, & \text{plastic loading} \end{cases} . \tag{17}$$

If the yield function indicates that the trial elastic stress is admissible ($f_{n+1}^{\text{trial}} \le 0$), the state variables are uploaded according to eq. (15) considering $\Delta\gamma = 0$. Otherwise, a correction of the state variables must be performed according to a return mapping algorithm. For linear isotropic hardening, the increment of the plastic multiplier is given in eq. (18). This value is then used in eq. (15) to update the state variables.

$$\Delta\gamma = \frac{f_{n+1}^{trial}}{E + E_p} \tag{18}$$

It is also necessary to calculate the tangent modulus, which is done as:

$$E_t = \frac{\partial\sigma}{\partial\varepsilon} = \begin{cases} E & \text{, in elastic range} \\ \dfrac{E\,E_p}{E + E_p} & \text{, in plastic range} \end{cases} . \tag{19}$$

Finally, these results are used to compute the natural forces with eq. (7). Numerical integrations are carried out along the element length and control cross-sections. Integration points along the longitudinal axis of the element define the control sections, where numerical integration is performed along their heights. The integration points along the height divide the cross-sections into layers. Figure 2 depicts this integration scheme. The components of the natural tangent stiffness matrix are the derivatives of the natural forces with respect to the natural displacements, as given by eq. (8), which leads to integral expressions that depend on the tangent modulus (not shown here). These components are evaluated with the same integration scheme of Fig. 2.

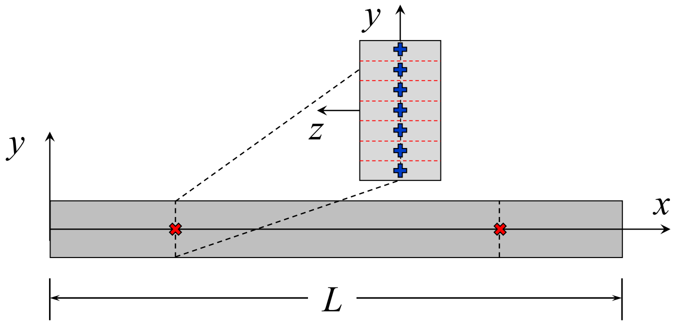

Figure 2. Integration points along the length of an element and the height of a control cross-section

3 Computational implementation

The implementation of the elasto-plastic material model required some OOP classes of the NUMA-TF program to be modified, and others to be added. In order to provide a general overview of the new code architecture, Fig. 3 presents the *Class Diagram* of the program following the UML (Unified Modeling Language) format [8]. The organization of the new classes and methods were inspired by the works of Menétrey and Zimmermann [3] and Commend and Zimmermann [4]. Among the main modifications is the transformation of the *Material* class into an abstract class, so that each material model is specified in a particular subclass. The attributes and methods of the new *Material* superclass are presented in Fig. 4, following the UML standard. All those methods were implemented in this work and are characteristic of finite element analyses of frame elements. Two material subclasses were created: *Material_LinearElastic* and *Material_ElastoPlastic*. For materials with linear-elastic behavior, the integrals that define the natural forces and the components of the natural tangent stiffness matrix of elements have an analytical solution and, therefore, there is no need to perform a numerical integration in this case. The *Material_LinearElastic* subclass is responsible to bring these analytical solutions. The *Material_ElastoPlastic*, on the other hand, directs to the numerical integration of the mentioned variables. This subclass is also an abstract class, so that different elasto-plastic models can be implemented. For example, the elasto-plastic behavior with the von Mises yield function, which is used when the stress state is not uniaxial and is already implemented (*Material_ElastoPlastic_VMises*), despite not being described in this paper.

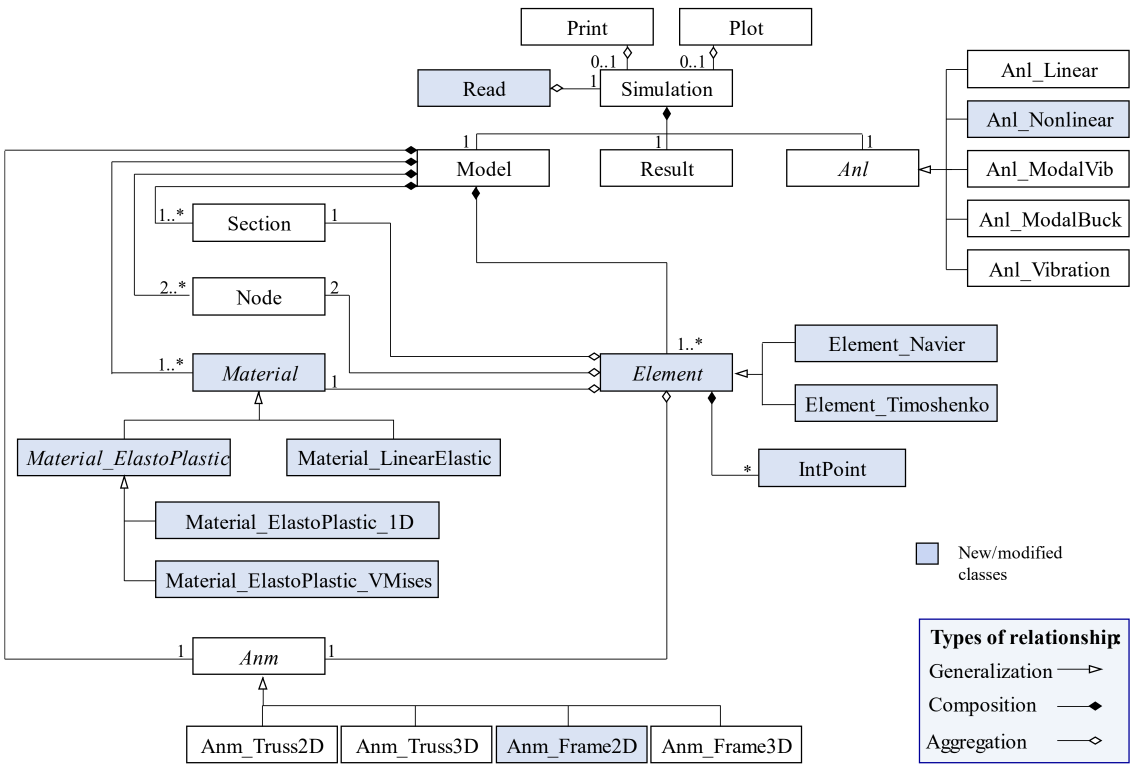

Figure 3. UML class diagram of NUMA-TF after the implementation of elasto-plastic constitutive model

Material	
+ id: int	+ G: float
+ E: float	+ rho: float
+ v: float	+ alpha: float
+ computeStresses()	
+ intLocalForceVct()	
+ getLocTangStiffMtrx_XY()	

Figure 4. UML representation of the new *Material* superclass

4 Numerical example

To validate the numerical implementation, the classical Lee frame (Fig. 3) was analyzed and its equilibrium path was compared against the solution provided by Battini [7]. Each bar was discretized into 10 elements. An elasto-plastic material was assumed: the Young's modulus is 7.2e6 kPa, the tangent modulus in plastic range is 7.2e5 kPa, and the initial yield stress is 10.44 kPa. The reference load, P, was taken as 2 kPa, and the model was analyzed up to a load ratio of $\lambda = 1$. To solve the nonlinear system of equations, the Cylindrical Arc Length method was used with a convergence tolerance of 1e-5, an initial load ratio increment of 0.01, and a target iterations number of 3 (to perform the incremental process with an adaptive increment size). The numerical integration to compute the natural forces and the components of the natural tangent matrix was performed with a Gauss-Legendre quadrature using two integration points along the elements length and seven integration points along the height of the control cross-sections.

Figure 6 presents the equilibrium paths obtained for this example. As it can be seen in Fig. 6a, the solution from the implementation of this work is in very good agreement with the reference work, which validates our solver. Figure 6b compares the equilibrium paths of the structure considering linear-elastic and elasto-plastic material models. The results indicate a considerable change in the equilibrium path, as the load ratio associated with the limit load values is decreased and the snap-back behavior is no longer present.

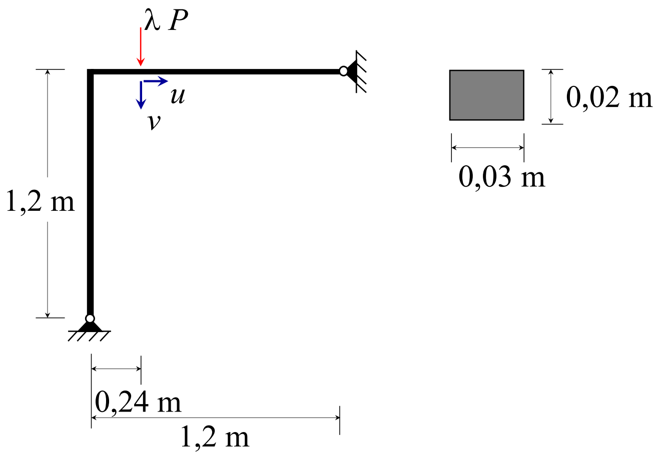

Figure 5. Lee frame model

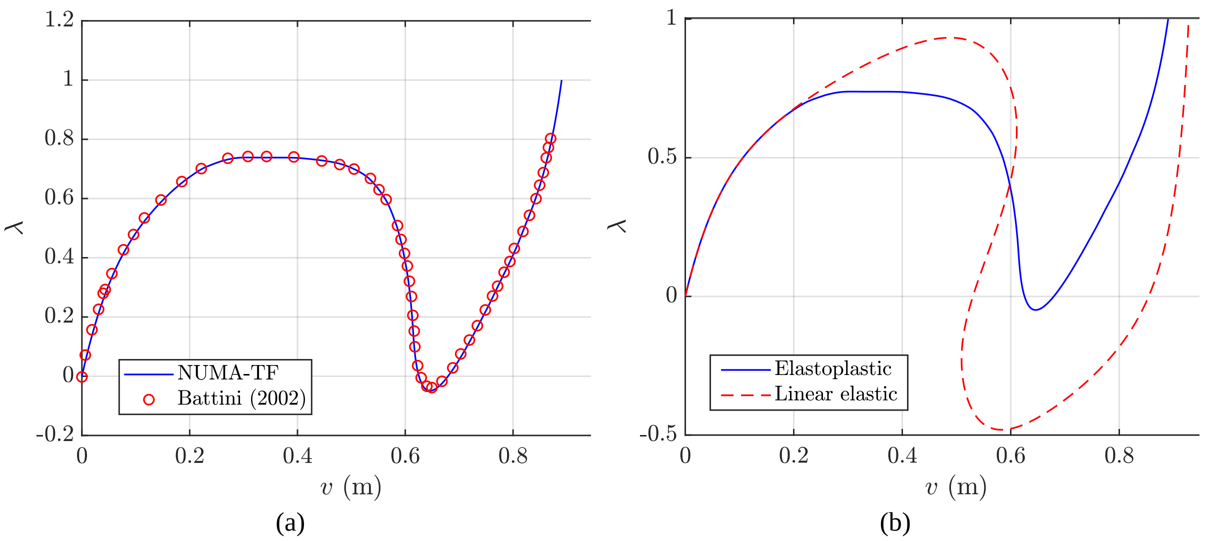

Figure 6. Lee frame results: (a) Validation and (b) comparison between linear-elastic and elasto-plastic responses

5 Conclusions

In this paper, material nonlinearity considering an elasto-plastic constitutive model was incorporated into a numerical solver of plane frame models and coupled with geometric nonlinearity. It was demonstrated that this coupling of nonlinear effects is straightforward with the corotational approach. A discussion about the computational implementation was also carried out. The development of a modular code structure based on the OOP proved to be very helpful for implementing new analysis features in the program. These implementations were then validated against the literature and the effects of considering the new type of nonlinearity was presented.

This work was limited to a specific type of element: two-dimensional frame element with flexural behavior described by Euler-Bernoulli theory. However, the program in which these developments took place can also deal with three-dimensional structures and Timoshenko theory for bending behavior. Therefore, this study is being continued to expand the inelastic material behavior for three-dimensional frames and trusses and for the consideration of shear deformation during bending of elements (Timoshenko theory). In fact, the implementation of the elasto-plastic behavior with the von Mises yield function is already targeting the inclusion of Timoshenko theory. The expansion to three-dimensional elements will be done by using quaternions to describe the spatial rotations in the corotational approach.

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References

[1] E. A. de Souza Neto, D. Peric and D. R. J. Owen. *Computational methods for plasticity: theory and applications*. John Wiley and Sons, 2008.

[2] R. Borst, M. A. Crisfield, J. J. C. Remmers and C. V. Verhoosel. *Non-linear Finite Element Analysis of Solids and Structures*. John Wiley and Sons, 2012.

[3] P. Menetrey and T. Zimmermann. "Object-oriented non-linear finite element analysis: Application to J2 plasticity". *Computers and Structures*, vol. 49, n. 5, p. 767-777, 1993.

[4] S. Commend and T. Zimmermann. "Object-oriented nonlinear finite element programming: a primer". *Advances in Engineering Software*, vol. 32, n. 8, p. 611-628, 2001.

[5] NUMA-TF (Numerical Analysis of Trusses and Frames): https://gitlab.com/rafaelrangel/numa-tf.

[6] R. L. Rangel. *Educational Tool for Structural Analysis of Plane Frame Models with Geometric Nonlinearity*. M.Sc. thesis, Pontifical Catholic University of Rio de Janeiro, 2019.

[7] J. M. Battini. *Co-rotational beam elements in instability problems*. Ph.D. thesis, Royal Institute of Technology, 2002.

[8] G. Booch, J. Rumbaugh and I. Jacobson. *The unified modeling language user guide*. Addison-Wesley, Upper Saddle River, NJ, 2005.