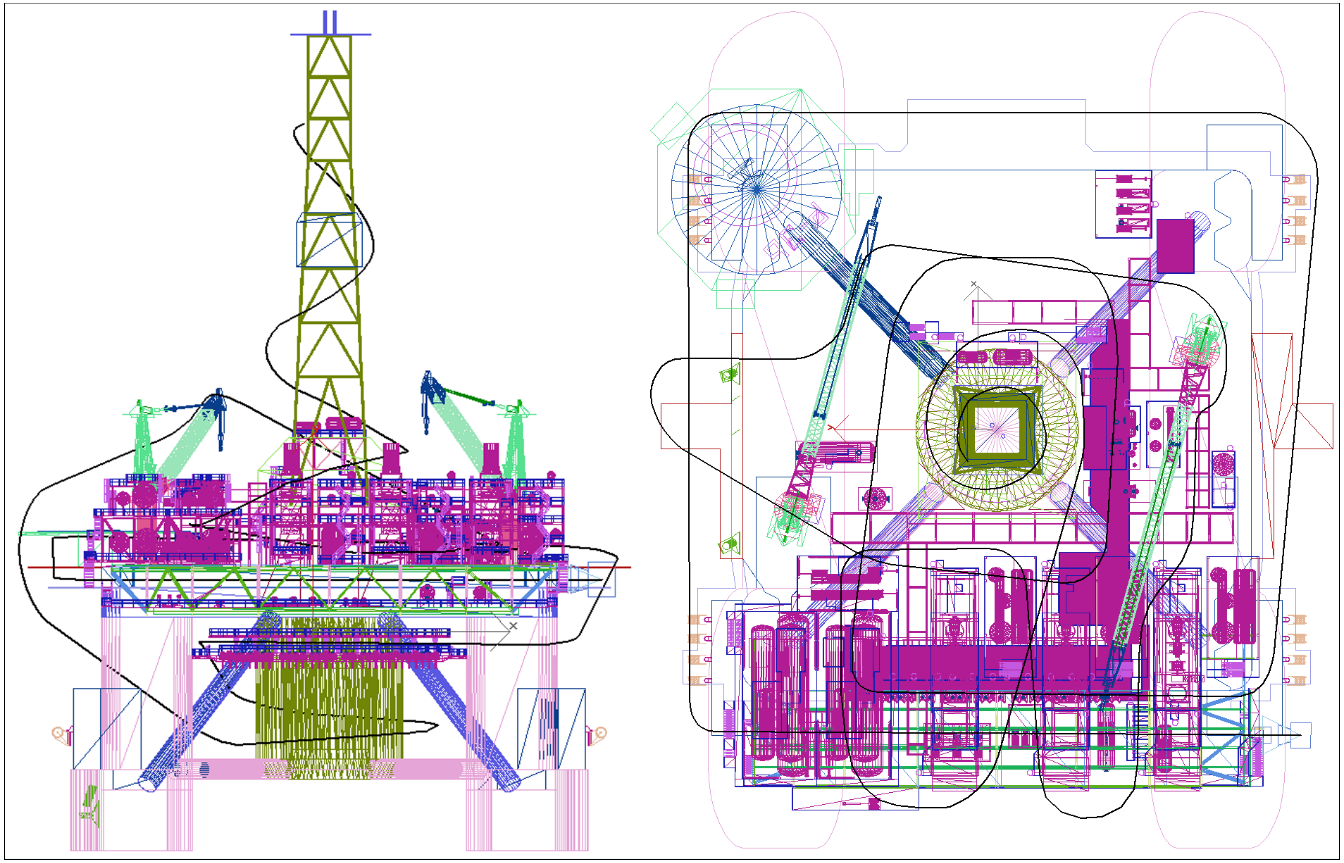
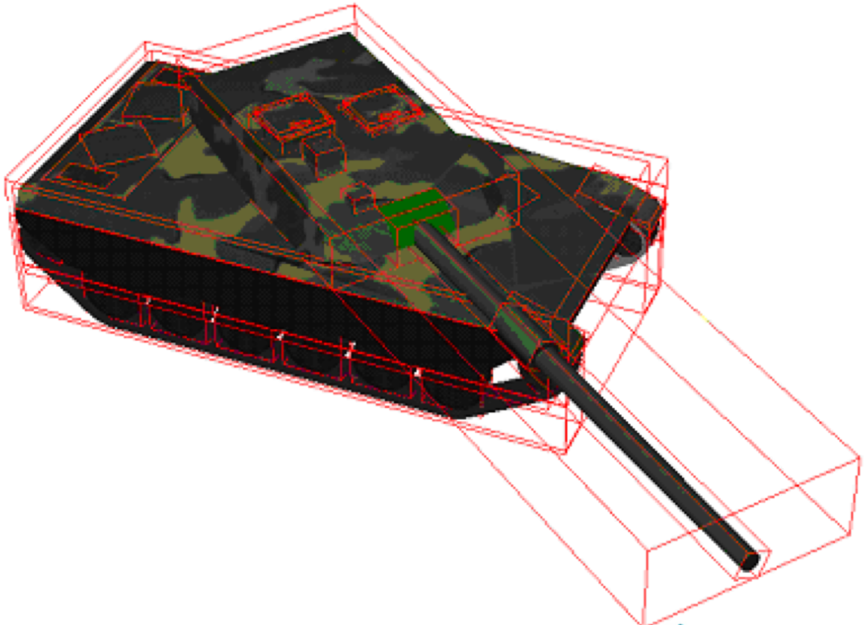
Transformações Geométricas

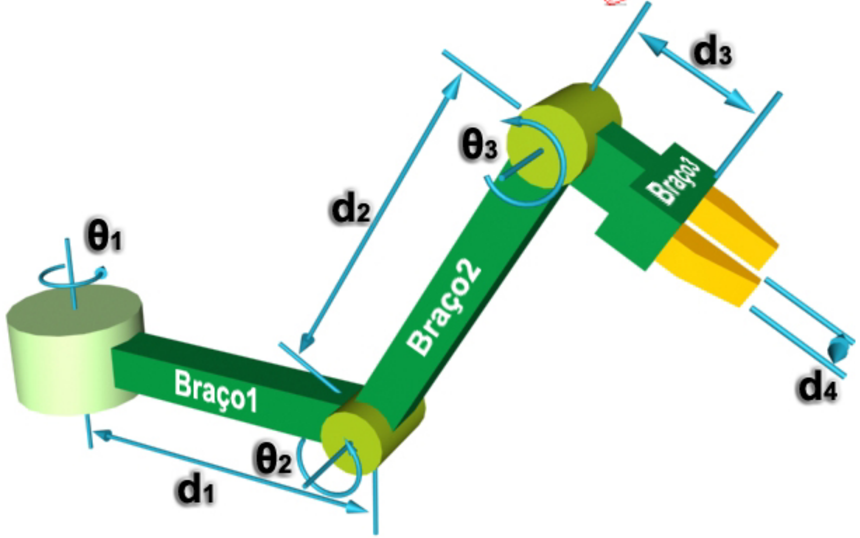
Coordenadas Homogêneas

Motivação: representação de movimentos e formas

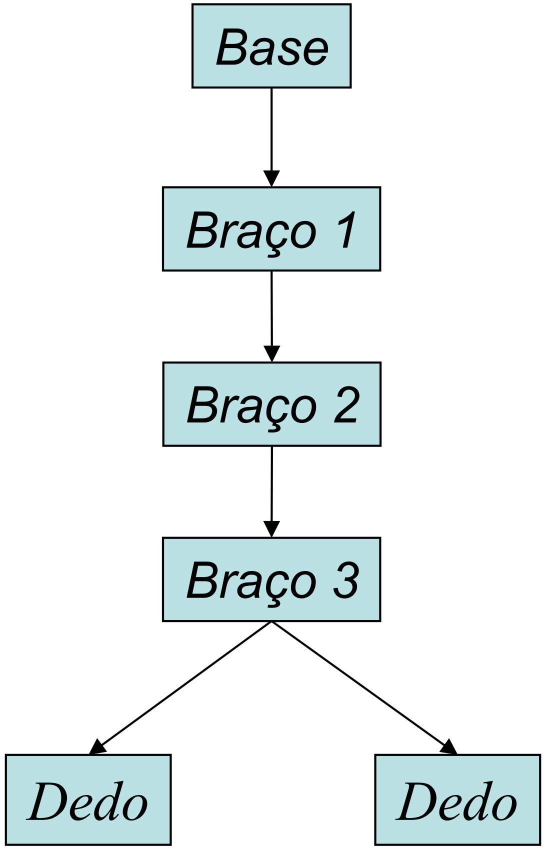


Objetos compostos hierarquicamente





Hieraquia de movimentos



Base

Braço 1

Braço 2

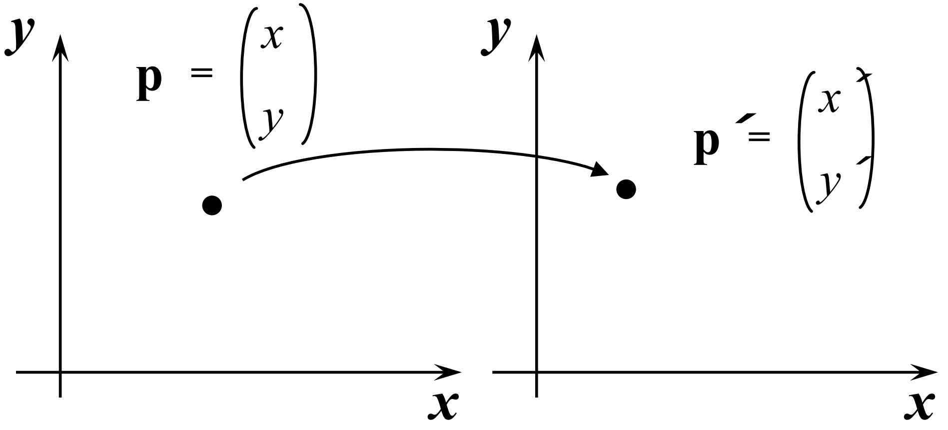
Braço 3

Dedo

Dedo

Hieraquia de transformações

Transformações $R^2 \rightarrow R^2$



$$\mathbf{p}' = T(\mathbf{p})$$

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = f\begin{pmatrix} x \\ y \end{pmatrix}$$

Exemplos:

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} x^2 + y \\ xy \end{pmatrix}$$

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} x\cos(y) \\ x\sin(y) \end{pmatrix}$$

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} 2x + 5y \\ x - y \end{pmatrix}$$

Transformações lineares $R^2 \rightarrow R^2$

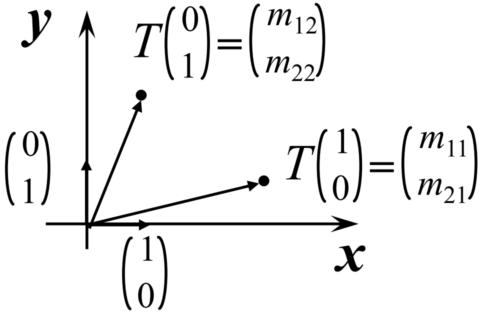
$$\boxed{T(a_1\mathbf{p}_1 + a_2\mathbf{p}_2) = a_1T(\mathbf{p}_1) + a_2T(\mathbf{p}_2), \quad a_i \in \mathbf{R}\ e\ \mathbf{p}_i \in \mathbf{R}^2}$$

Mostre que:

A) $T(\mathbf{0}) = \mathbf{0}$

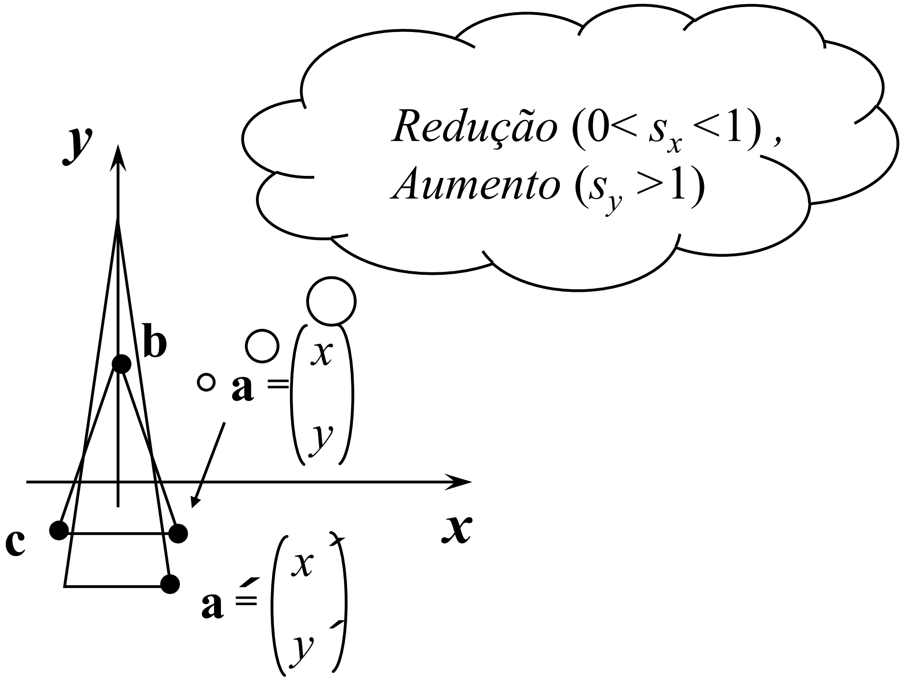
$$T(0) = T(\mathbf{p} - \mathbf{p}) = T(\mathbf{p}) - T(\mathbf{p}) = 0$$

B)



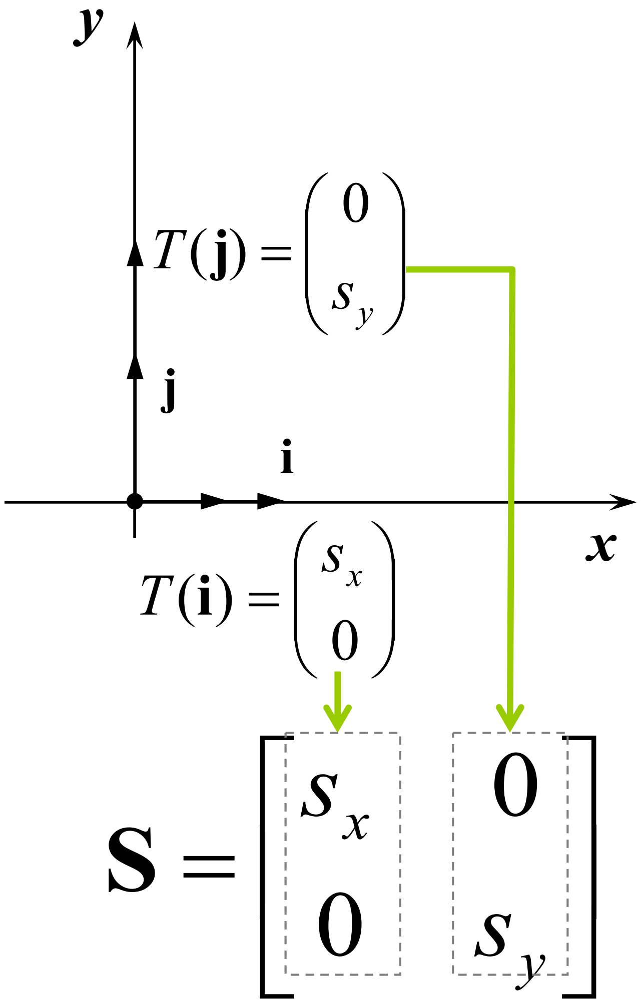
$$\Leftrightarrow \qquad \begin{pmatrix}x'\\y'\end{pmatrix} = \begin{bmatrix}m_{11} & m_{12}\\ m_{21} & m_{22}\end{bmatrix}\begin{pmatrix}x\\y\end{pmatrix}$$

Transformações lineares: escala



Redução $(0< s_x <1)$,
Aumento $(s_y >1)$

$$\begin{pmatrix}x'\\y'\end{pmatrix}=\begin{pmatrix}s_x x\\s_y y\end{pmatrix}=\begin{bmatrix}s_x&0\\0&s_y\end{bmatrix}\begin{pmatrix}x\\y\end{pmatrix}$$

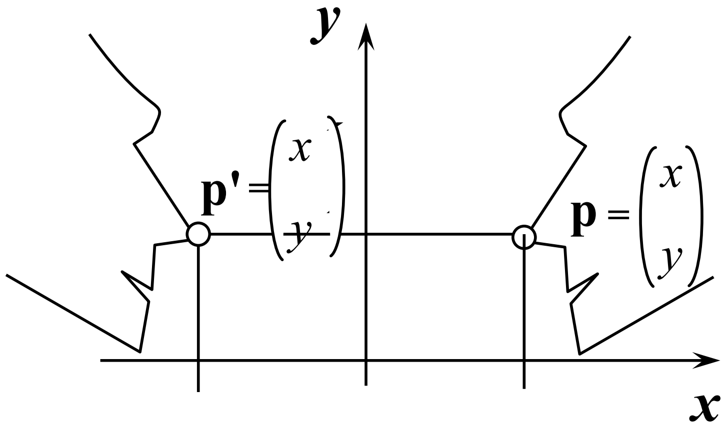


$$T(\mathbf{j}) = \begin{pmatrix}0\\s_y\end{pmatrix}$$

$$T(\mathbf{i}) = \begin{pmatrix}s_x\\0\end{pmatrix}$$

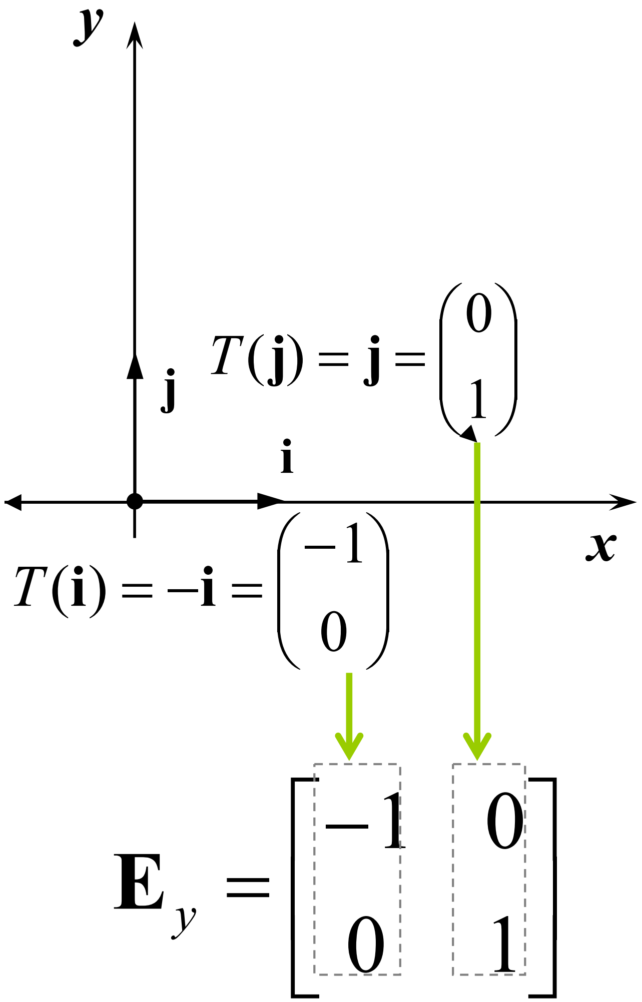
$$\mathbf{S} = \begin{bmatrix}s_x&0\\0&s_y\end{bmatrix}$$

Transformações lineares: espelhamento



$$x´ = -1x$$
$$y´ = \quad y$$

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} -x \\ y \end{pmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

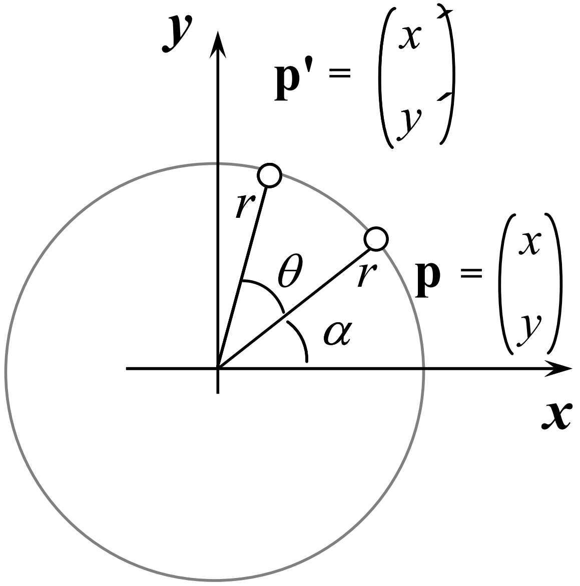


$$T(\mathbf{j}) = \mathbf{j} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$T(\mathbf{i}) = -\mathbf{i} = \begin{pmatrix} -1 \\ 0 \end{pmatrix}$$

$$\mathbf{E}_y = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$$

Transformações lineares: rotação



$$\sin(\alpha+\theta) = \sin\alpha\cdot\cos\theta + \cos\alpha\cdot\sin\theta$$

$$\cos(\alpha+\theta) = \cos\alpha\cdot\cos\theta - \sin\alpha\cdot\sin\theta$$

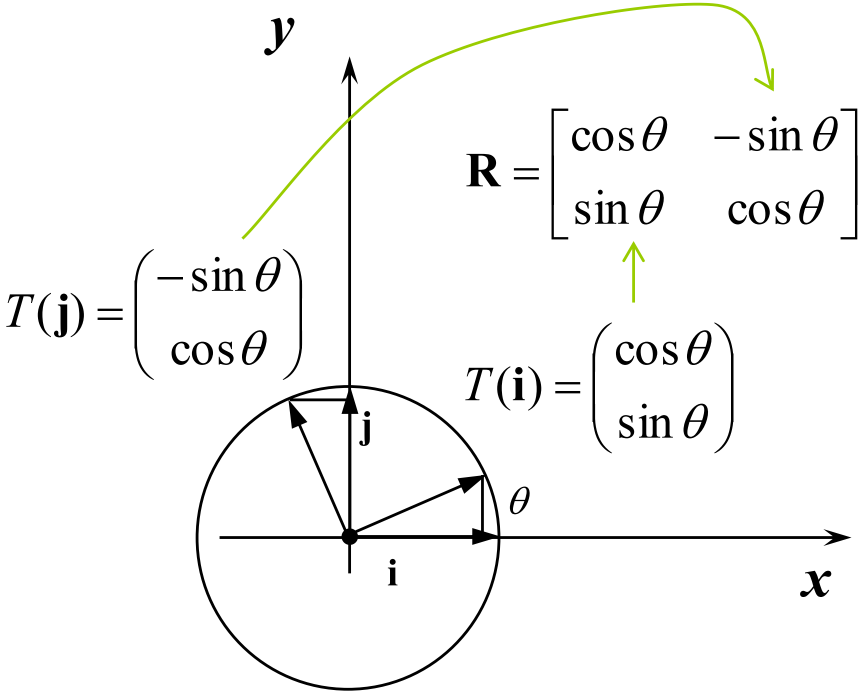
$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} r\cos(\alpha+\theta) \\ r\sin(\alpha+\theta) \end{pmatrix} = \begin{pmatrix} r\cos\alpha\cdot\cos\theta - r\sin\alpha\sin\theta \\ r\sin\alpha\cdot\cos\theta + r\cos\alpha\cdot\sin\theta \end{pmatrix}$$

$x´ = x.cos\ \theta - y.sen\ \theta$

$y´ = x.sen\ \theta + y.cos\ \theta$

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

Transformações Lineares: matriz derivada pela geometria



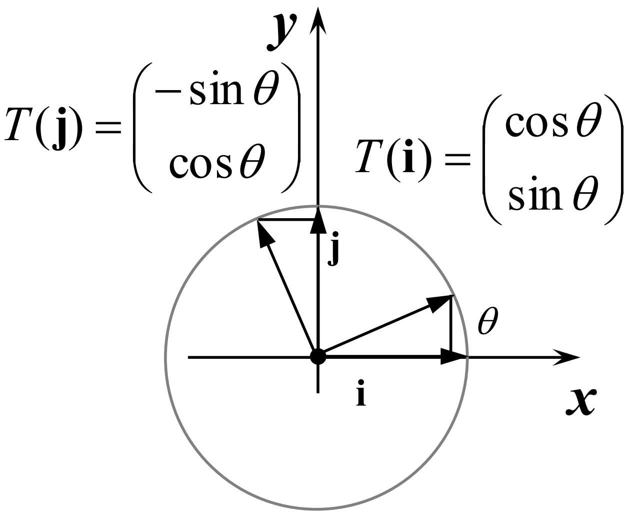
$$T(\mathbf{j}) = \begin{pmatrix} -\sin\theta \\ \cos\theta \end{pmatrix}$$

$$\mathbf{R} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

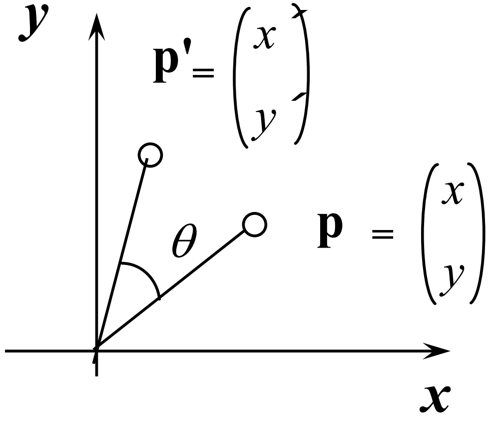
$$T(\mathbf{i}) = \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix}$$

$$\mathbf{R} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

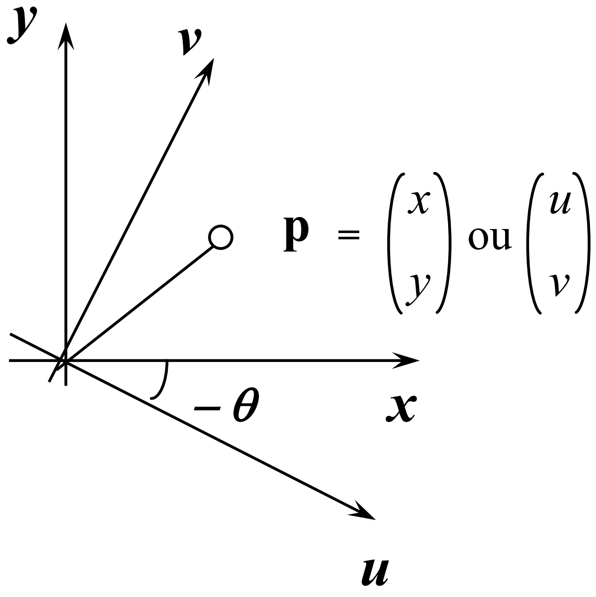
Mudança de referêncial



$$T(\mathbf{j}) = \begin{pmatrix} -\sin\theta \\ \cos\theta \end{pmatrix} \quad T(\mathbf{i}) = \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix}$$



$$\mathbf{p'} = \begin{pmatrix} x' \\ y' \end{pmatrix} \quad \mathbf{p} = \begin{pmatrix} x \\ y \end{pmatrix}$$



$$\mathbf{p} = \begin{pmatrix} x \\ y \end{pmatrix} \text{ ou } \begin{pmatrix} u \\ v \end{pmatrix}$$

$$\mathbf{R} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

$$\begin{pmatrix} u \\ v \end{pmatrix} = \begin{bmatrix} cos\ \theta & \text{-}sen\ \theta \\ sen\ \theta & cos\ \theta \end{bmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

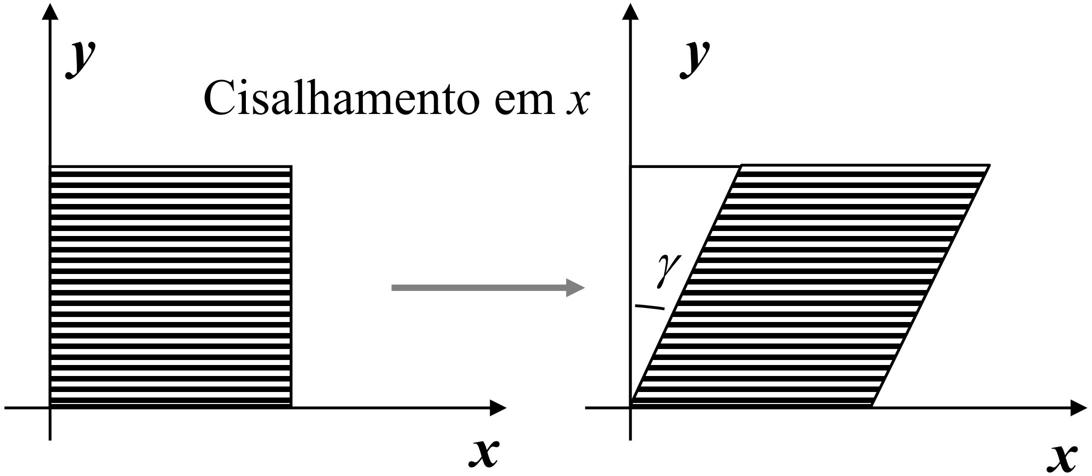
$$\begin{pmatrix} u \\ v \end{pmatrix} = \begin{bmatrix} u_x & u_y \\ v_x & v_y \end{bmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

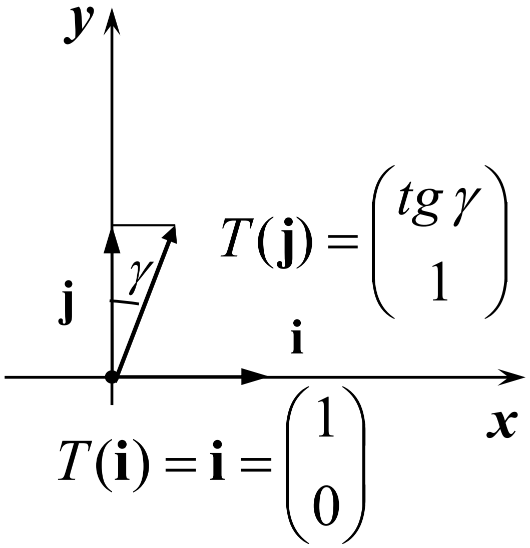
Para montarmos a matriz que transforma as coordenadas de um refencial *xy* para um novo refencial *uv* basta escrevermos as linhas como sendo os unitários das direções.

Mudança de coordenadas entre sistemas rotacionados

- As coordenas de um ponto rodado de um ângulo em relação a um sistema são iguais as coordenadas do ponto original em relação a um sistema que sofre a rotação **inversa**.
- Como o novo sistema sofre a rotação inversa, a matriz de rotação é a inversa da matriz que levaria da base original para a este novo sistema.
- As colunas de uma matriz de uma rotação são as transformadas dos vetores da base e a transposta desta matriz é a sua inversa (rotação⇒matriz ortonormal).
- Logo as **linhas** da matriz que escreve uma mudança entre bases ortonormais rodadas são as coordenadas do vetores da nova base em relação a base original.

Transformações lineares: cisalhamento (shear)





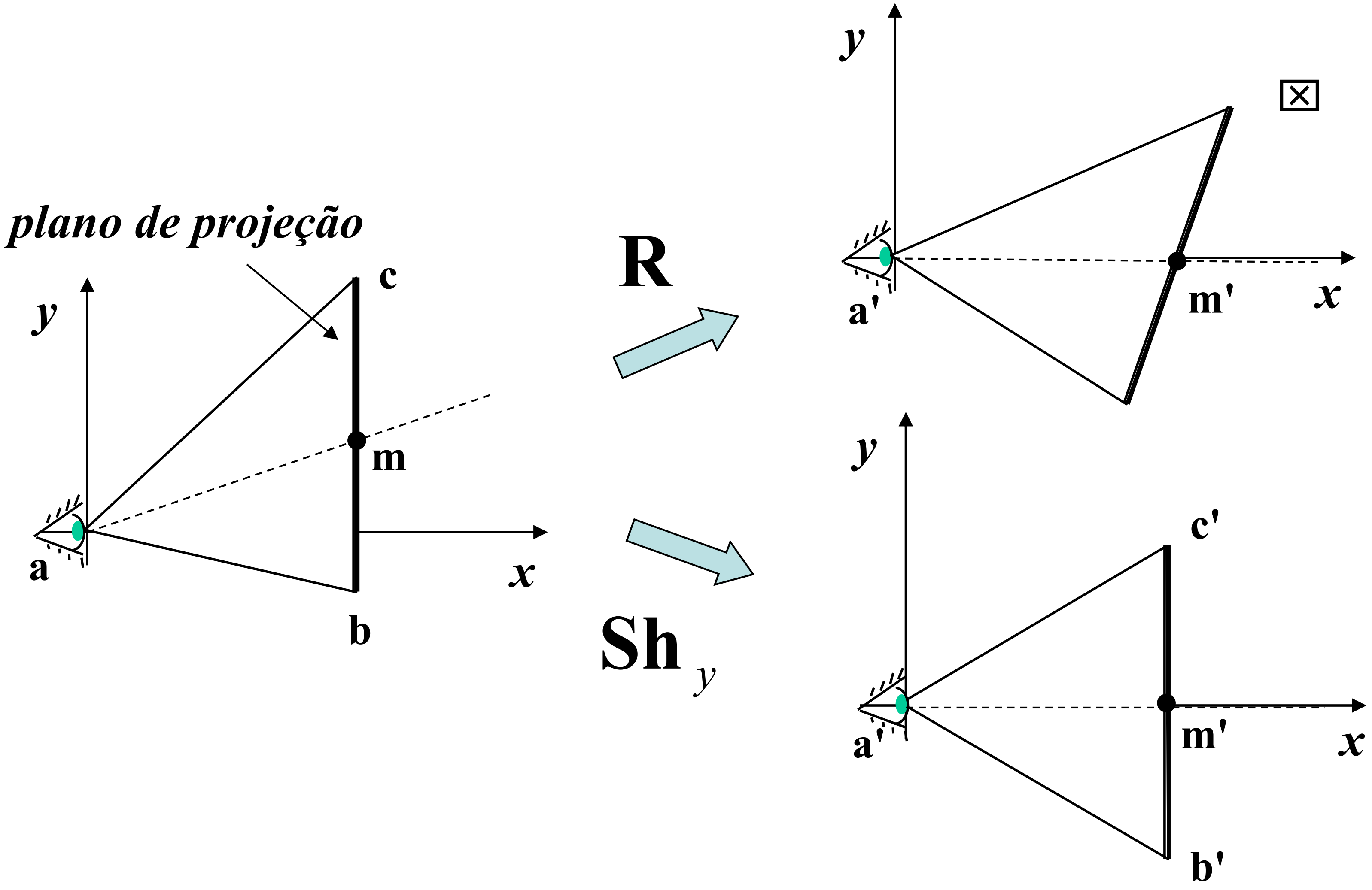
$$T(\mathbf{j}) = \begin{pmatrix} tg\,\gamma \\ 1 \end{pmatrix}$$

$$T(\mathbf{i}) = \mathbf{i} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

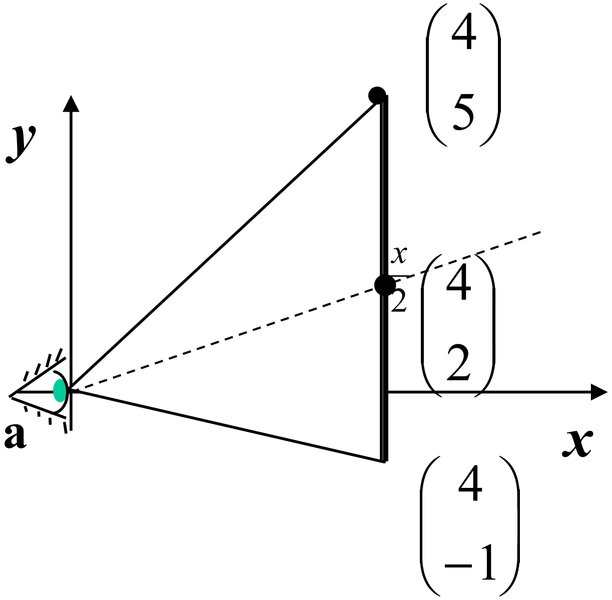
$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} x + y\tan\gamma \\ y \end{pmatrix} = \begin{bmatrix} 1 & \tan\gamma \\ 0 & 1 \end{bmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

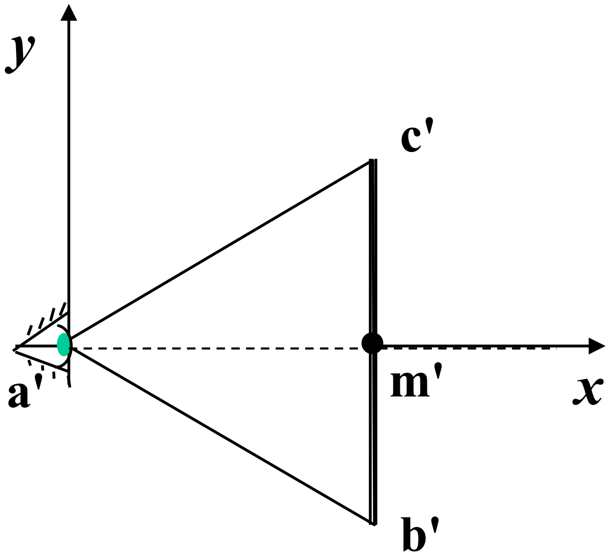
$$Sh_x = \begin{bmatrix} 1 & tg\,\gamma \\ 0 & 1 \end{bmatrix}$$

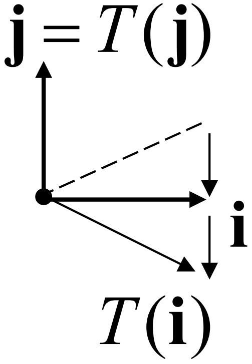
Exemplo de aplicação do cisalhamento



Exemplo de aplicação do cisalhamento





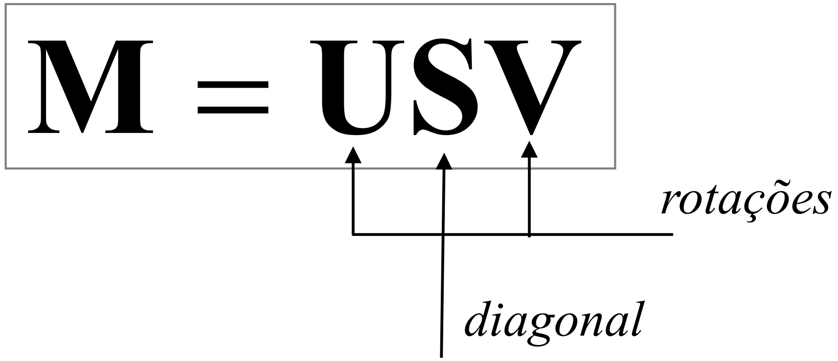


$$x' = x$$

$$y' = y - \frac{x}{2}$$

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} x \\ -x/2 + y \end{pmatrix} = \begin{bmatrix} 1 & 0 \\ -1/2 & 1 \end{bmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

Decomposição Singular de Matrizes



$$\boxed{\mathbf{M} = \mathbf{USV}}$$

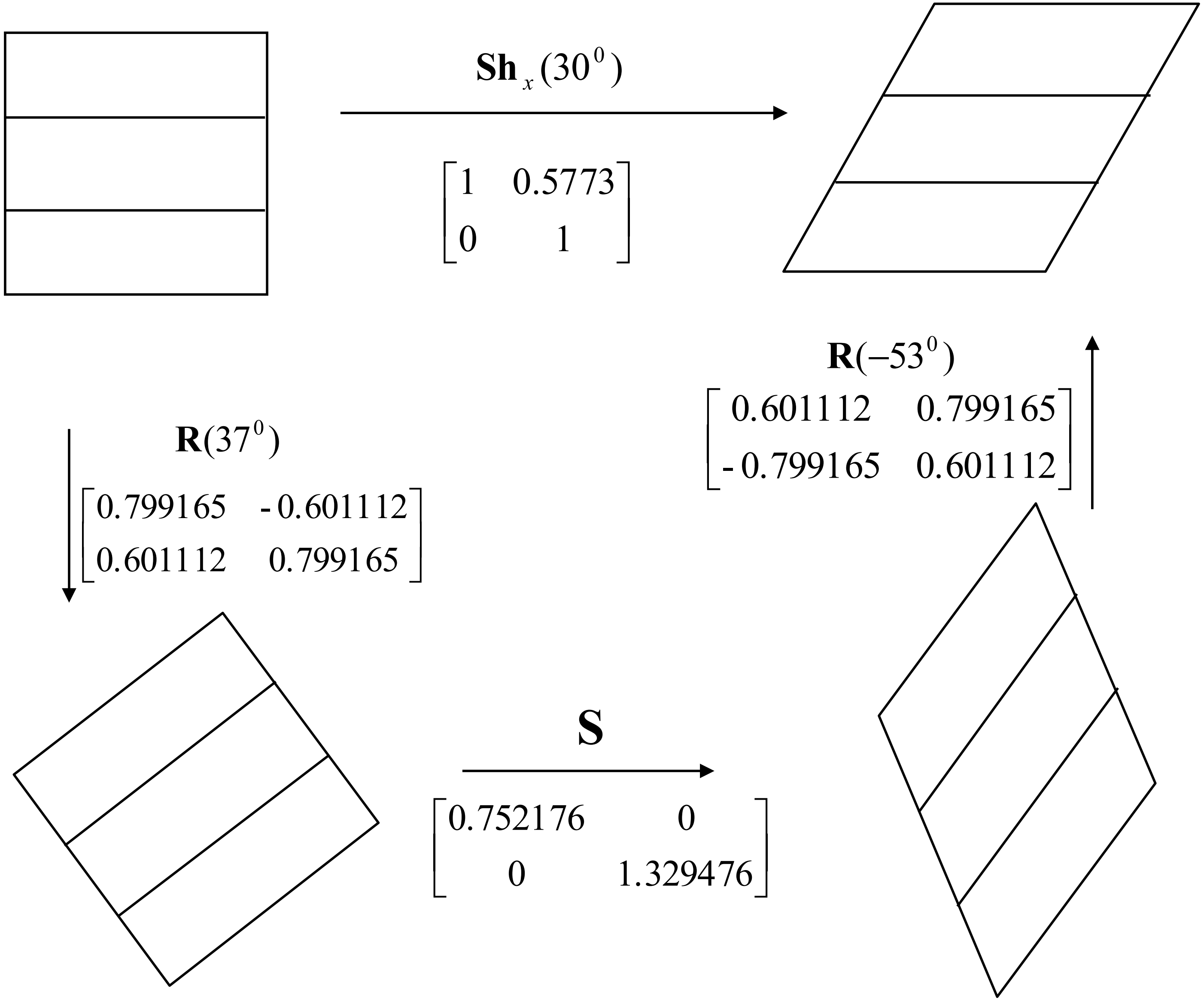
$$\mathbf{M} = \begin{bmatrix} 1 & \tan 30^0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0.5773 \\ 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0.5773 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0.601112 & 0.799165 \\ -0.799165 & 0.601112 \end{bmatrix} \begin{bmatrix} 0.752176 & 0 \\ 0 & 1.329476 \end{bmatrix} \begin{bmatrix} 0.799165 & -0.601112 \\ 0.601112 & 0.799165 \end{bmatrix}$$

$$\begin{bmatrix} 1 & \tan 30^0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} \cos(-53^0) & -\sin(-53^0) \\ \sin(-53^0) & \cos(-53^0) \end{bmatrix} \begin{bmatrix} 0.752176 & 0 \\ 0 & 1.329476 \end{bmatrix} \begin{bmatrix} \cos(37^0) & -\sin(37^0) \\ \sin(37^0) & \cos(37^0) \end{bmatrix}$$

$$\mathbf{Sh}_x(30^0) = \mathbf{R}(-53^0)\mathbf{S}(0.752,1.329)\mathbf{R}(37^0)$$

Exemplo: cisalhamento como composição de rotações e escala



$\mathbf{Sh}_x(30^0)$

$$\begin{bmatrix} 1 & 0.5773 \\ 0 & 1 \end{bmatrix}$$

$\mathbf{R}(37^0)$

$$\begin{bmatrix} 0.799165 & -0.601112 \\ 0.601112 & 0.799165 \end{bmatrix}$$

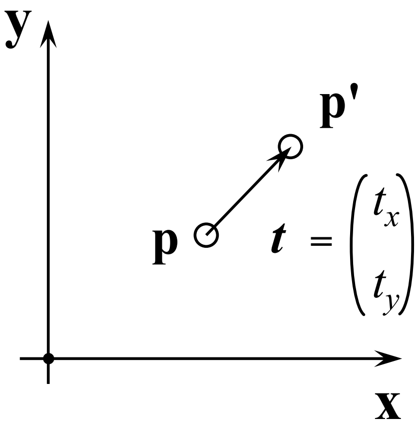
$\mathbf{S}$

$$\begin{bmatrix} 0.752176 & 0 \\ 0 & 1.329476 \end{bmatrix}$$

$\mathbf{R}(-53^0)$

$$\begin{bmatrix} 0.601112 & 0.799165 \\ -0.799165 & 0.601112 \end{bmatrix}$$

Transformações Geométricas: Translação



$$\boldsymbol{t} = \begin{pmatrix} t_x \\ t_y \end{pmatrix}$$

$$\mathbf{p'} = \begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} t_x \\ t_y \end{pmatrix}$$

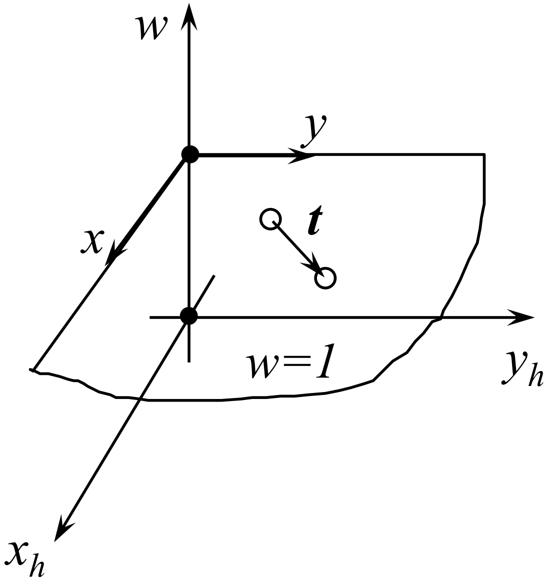
$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{bmatrix} ? & ? \\ ? & ? \end{bmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

⇦ *Não pode ser escrito na forma* ☒

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} t_x \\ t_y \end{pmatrix}$$

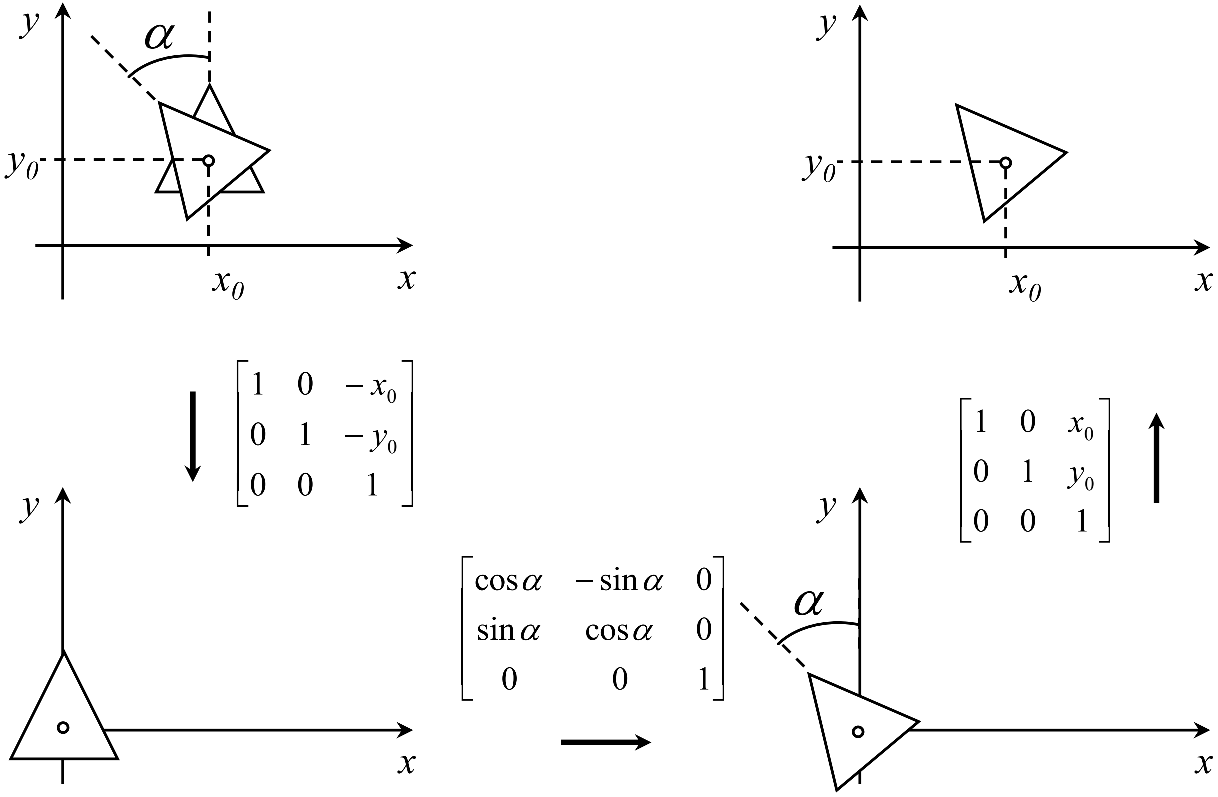
⇦ *Ruim para implementação* ☒

Translação num plano do R^3



$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & 0 & t_x \\ 0 & 1 & t_y \\ 0 & 0 & 1 \end{bmatrix}}_{\textit{matriz de translação}} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

Concatenação



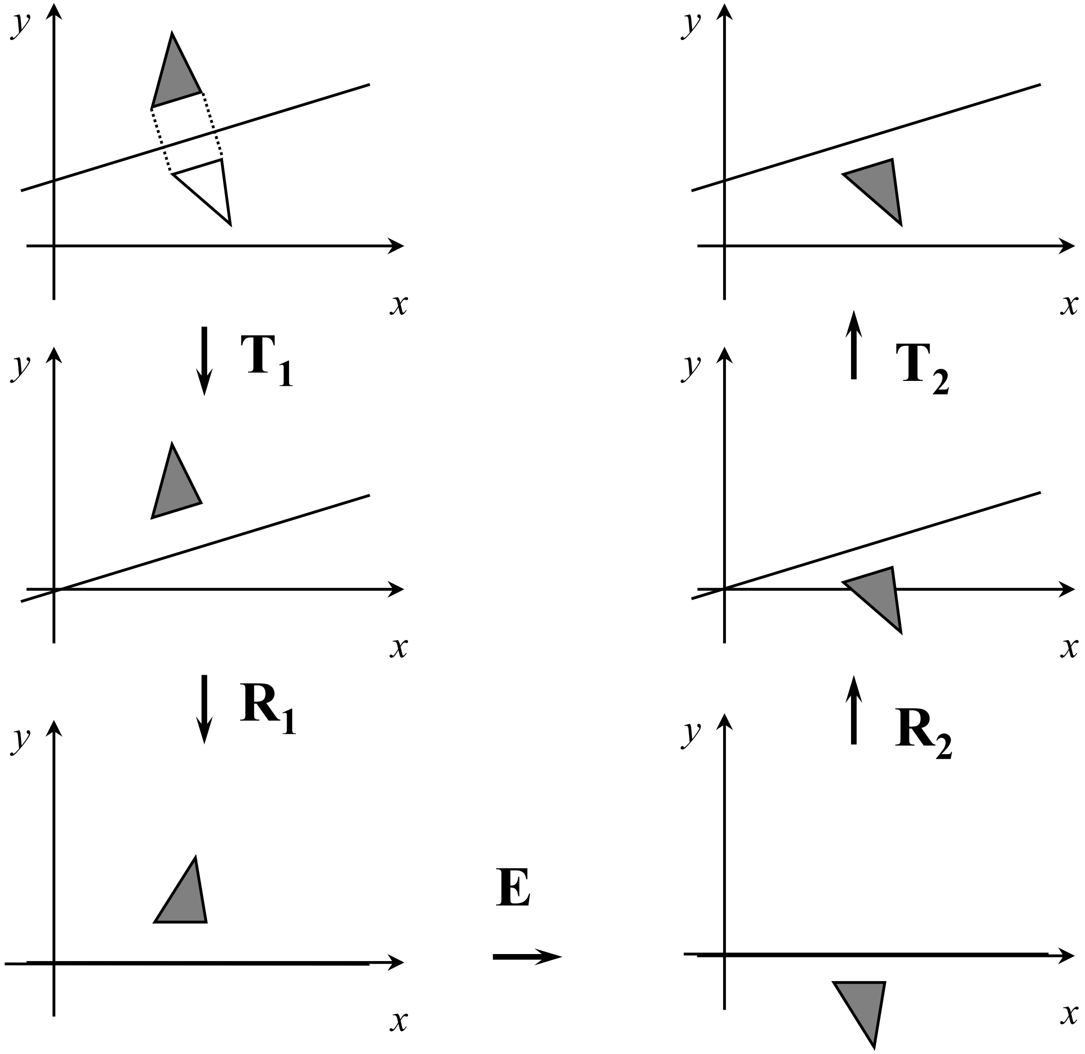
$$\begin{bmatrix} 1 & 0 & -x_0 \\ 0 & 1 & -y_0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} \cos\alpha & -\sin\alpha & 0 \\ \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & x_0 \\ 0 & 1 & y_0 \\ 0 & 0 & 1 \end{bmatrix}$$

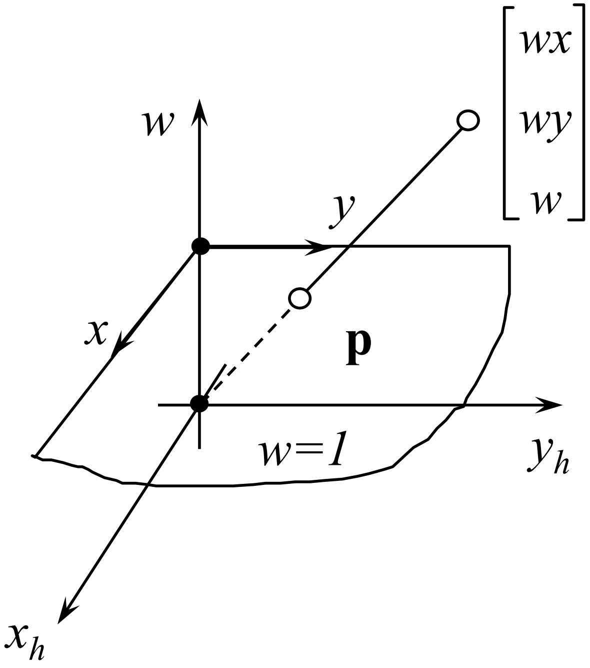
$$\boxed{\begin{Bmatrix} x' \\ y' \\ 1 \end{Bmatrix} = \begin{bmatrix} 1 & 0 & x_0 \\ 0 & 1 & y_0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos\alpha & -\sin\alpha & 0 \\ \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & -x_0 \\ 0 & 1 & -y_0 \\ 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} x \\ y \\ 1 \end{Bmatrix}}$$

Concatenação



$$\boxed{P' = T_2 \, R_2 \, E \, R_1 \, T_1 \, P}$$

Coordenadas projetivas (ou homogêneas)



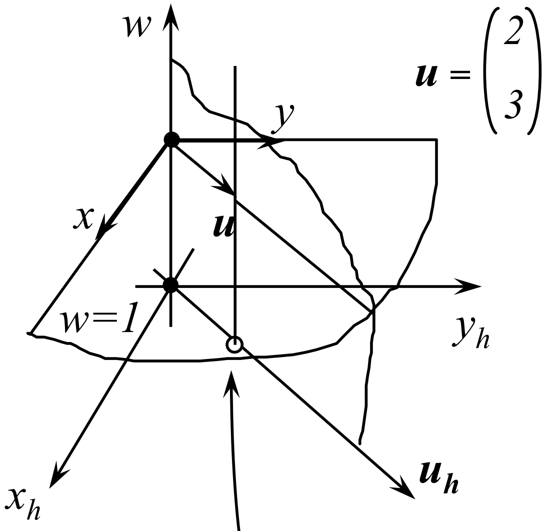
$$\mathbf{p} = \begin{pmatrix} x \\ y \end{pmatrix} \triangleq \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} \triangleq \begin{bmatrix} wx \\ wy \\ w \end{bmatrix} = \begin{bmatrix} x_h \\ y_h \\ w \end{bmatrix}$$

$$\boxed{\begin{array}{ll} x = x_h/w & \\ & w>0 \\ y = y_h/w & \end{array}}$$

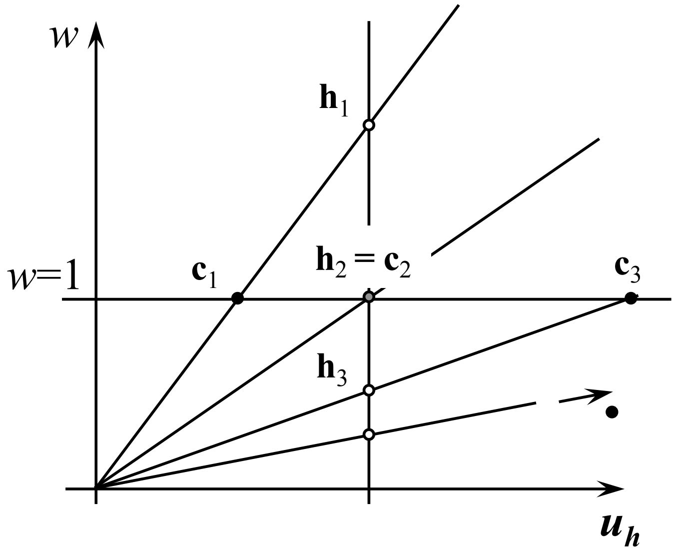
Ex.:

$$\begin{pmatrix} 3 \\ 2 \end{pmatrix} \triangleq \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix} \triangleq \begin{bmatrix} 6 \\ 4 \\ 2 \end{bmatrix} = \begin{bmatrix} 9 \\ 6 \\ 3 \end{bmatrix}$$

Vantagens das coordenadas homogêneas (pontos no infinito)



$$\begin{bmatrix} 2 \\ 3 \\ 0 \end{bmatrix} = \begin{pmatrix} ? \\ ? \end{pmatrix}$$

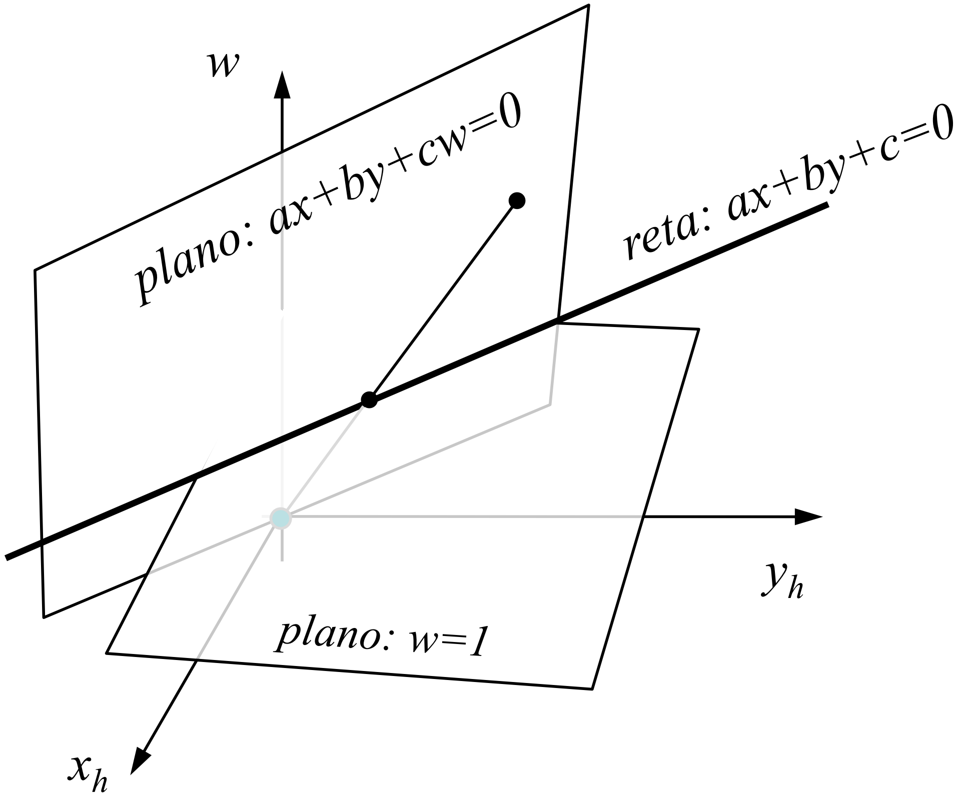


$$\begin{array}{ccccc} \mathbf{h}_1 & \mathbf{h}_2 & \mathbf{h}_3 & \mathbf{h}_4 & \\ \begin{bmatrix} 2 \\ 3 \\ 2 \end{bmatrix} & \begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix} & \begin{bmatrix} 2 \\ 3 \\ 1/2 \end{bmatrix} & \begin{bmatrix} 2 \\ 3 \\ 1/4 \end{bmatrix} \ldots & \begin{bmatrix} 2 \\ 3 \\ 0 \end{bmatrix} \end{array}$$

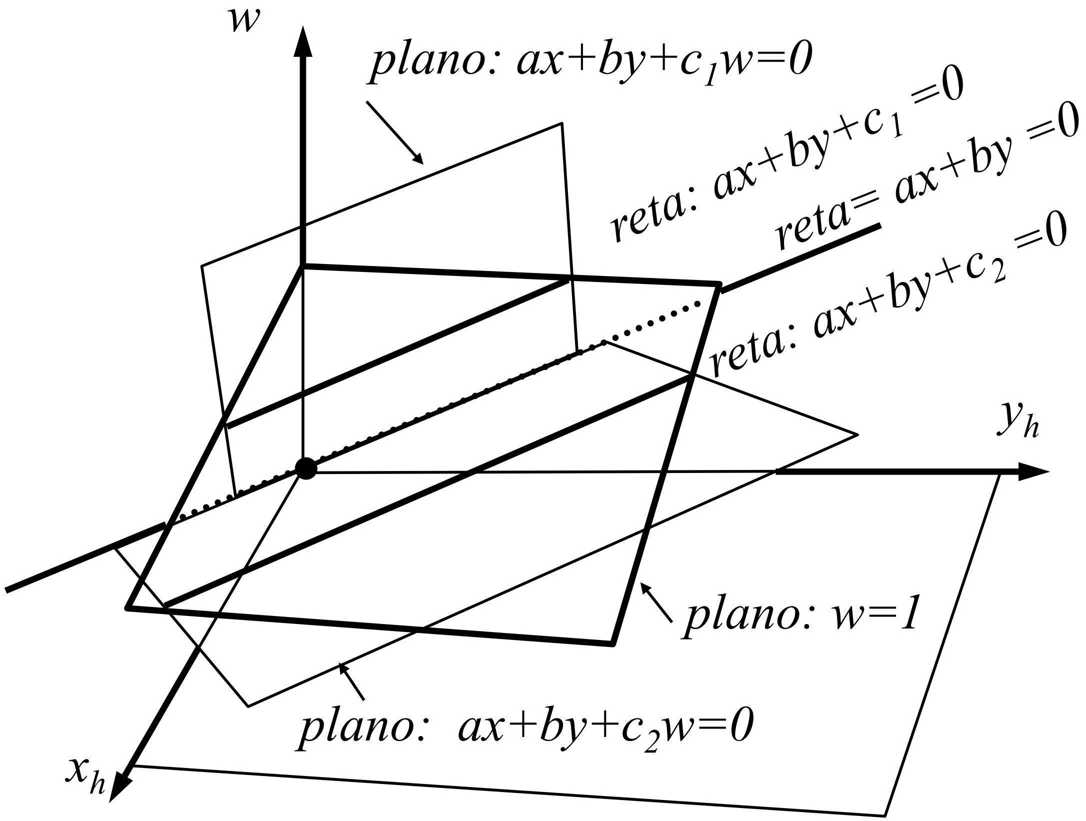
$$\begin{array}{cccc} \mathbf{c}_1 & \mathbf{c}_2 & \mathbf{c}_3 & \mathbf{c}_4 \\ \begin{pmatrix} 1 \\ 1.5 \end{pmatrix} & \begin{pmatrix} 2 \\ 3 \end{pmatrix} & \begin{pmatrix} 4 \\ 6 \end{pmatrix} & \begin{pmatrix} 8 \\ 12 \end{pmatrix} \end{array}$$

infinito
na
direção
(2,3)

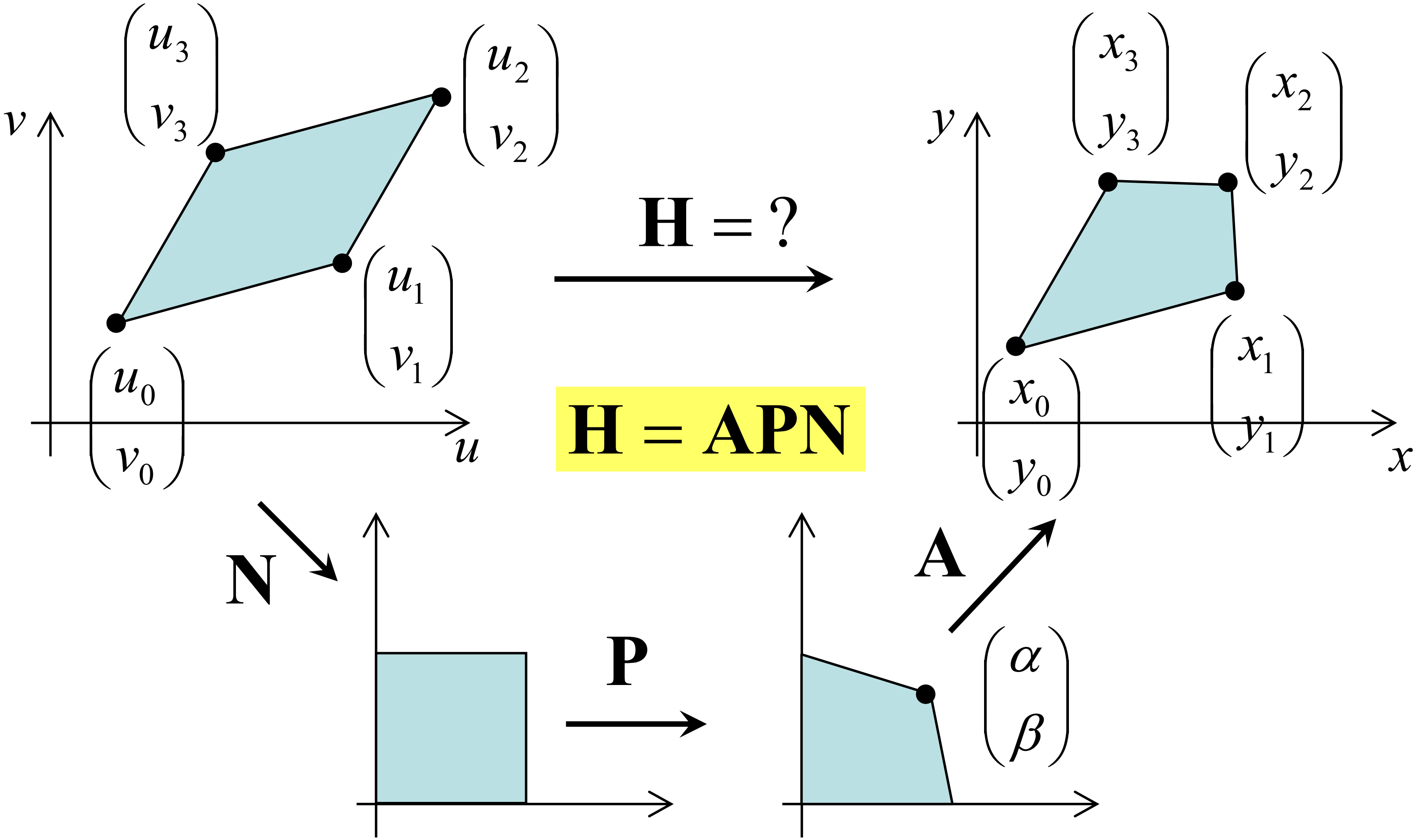
Reta no espaço projetivo



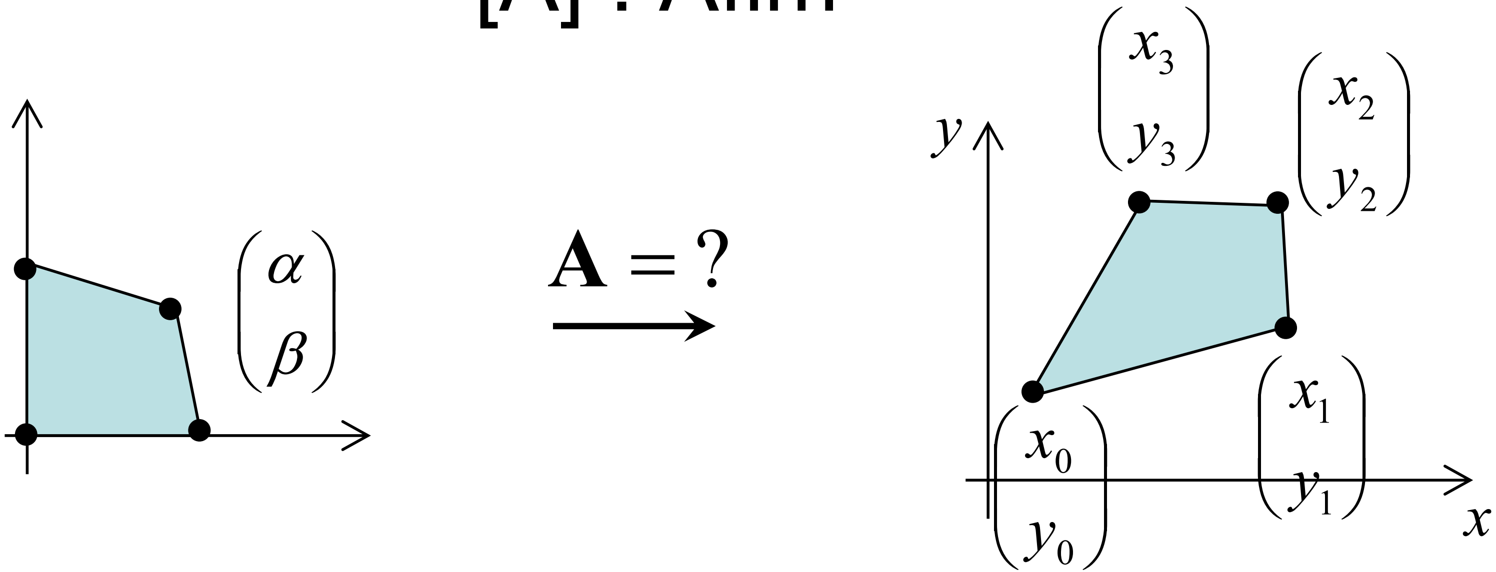
Reta paralelas no espaço projetivo



Matriz da Homografia



[A] : Afim

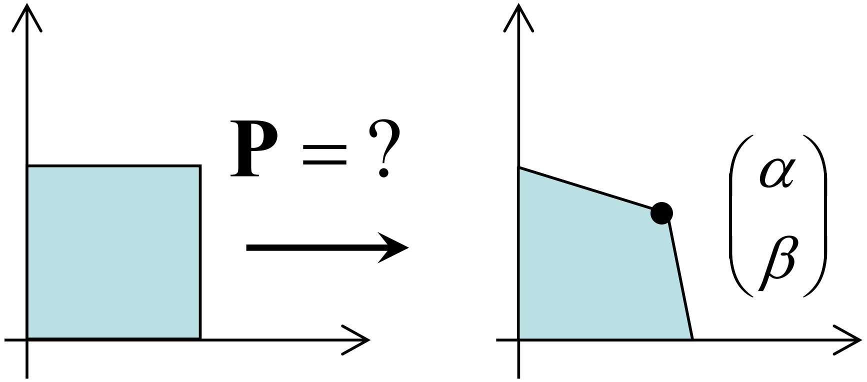


$$\mathbf{A} = \begin{bmatrix} x_1 - x_0 & x_3 - x_0 & x_0 \\ y_1 - y_0 & y_3 - y_0 & y_0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\begin{pmatrix}\alpha\\ \beta\end{pmatrix} = \mathbf{A}^{-1}\begin{pmatrix}x_2\\ y_2\\ 1\end{pmatrix} = \begin{bmatrix} x_1 - x_0 & x_3 - x_0 & x_0 \\ y_1 - y_0 & y_3 - y_0 & y_0 \\ 0 & 0 & 1 \end{bmatrix}^{-1}\begin{pmatrix}x_2\\ y_2\\ 1\end{pmatrix}$$

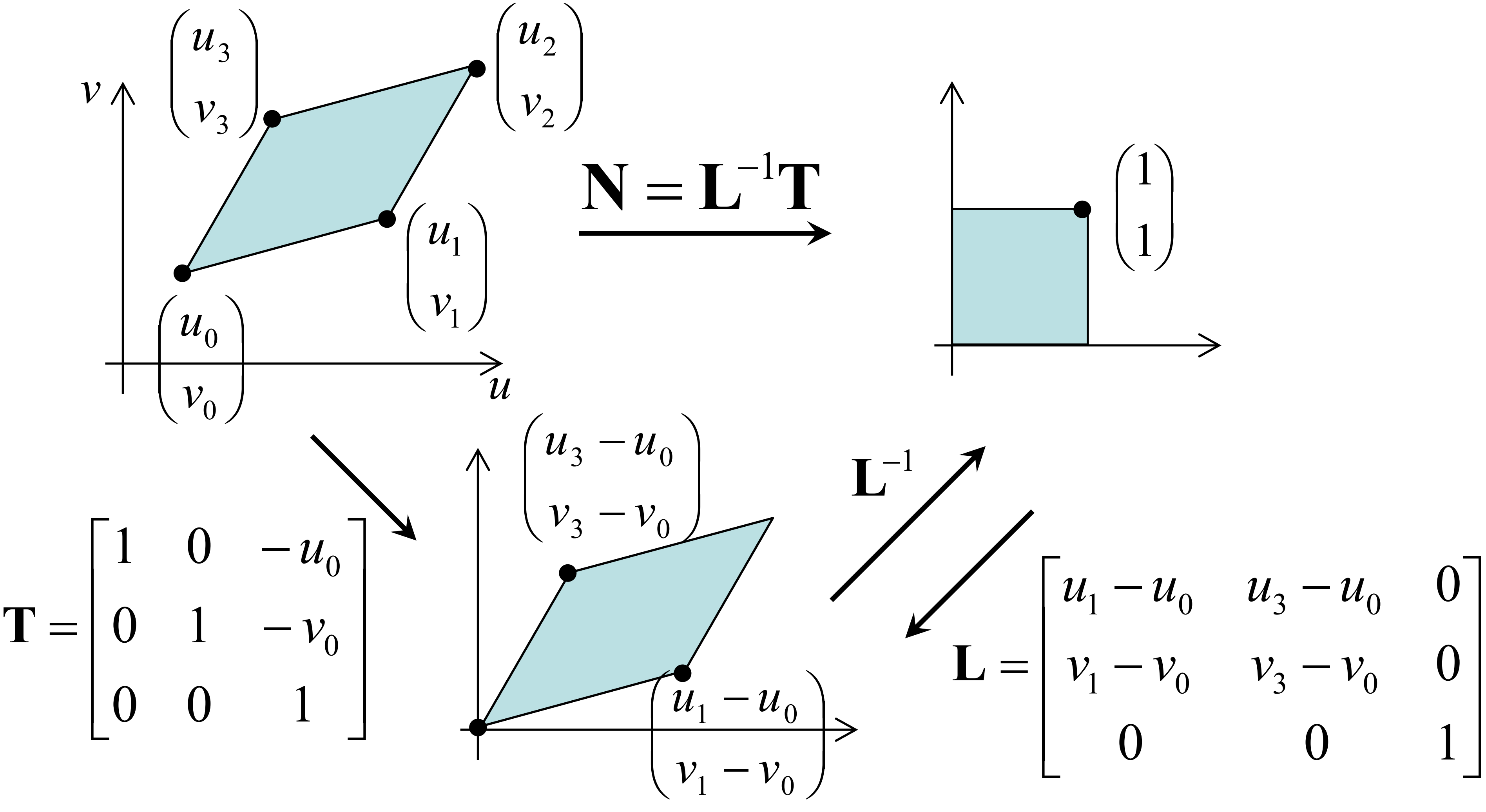
Obs: Se fosse um paralelograma a imagem do ponto 2 seria $(1,1)^T$ e não $(\alpha, \beta)^T$

[P] : Projetiva



$$\mathbf{P} = \begin{bmatrix} \alpha & 0 & 0 \\ 0 & \beta & 0 \\ 1-\beta & 1-\alpha & \alpha+\beta-1 \end{bmatrix}$$

[N] : Paralelograma para quadrado unitário

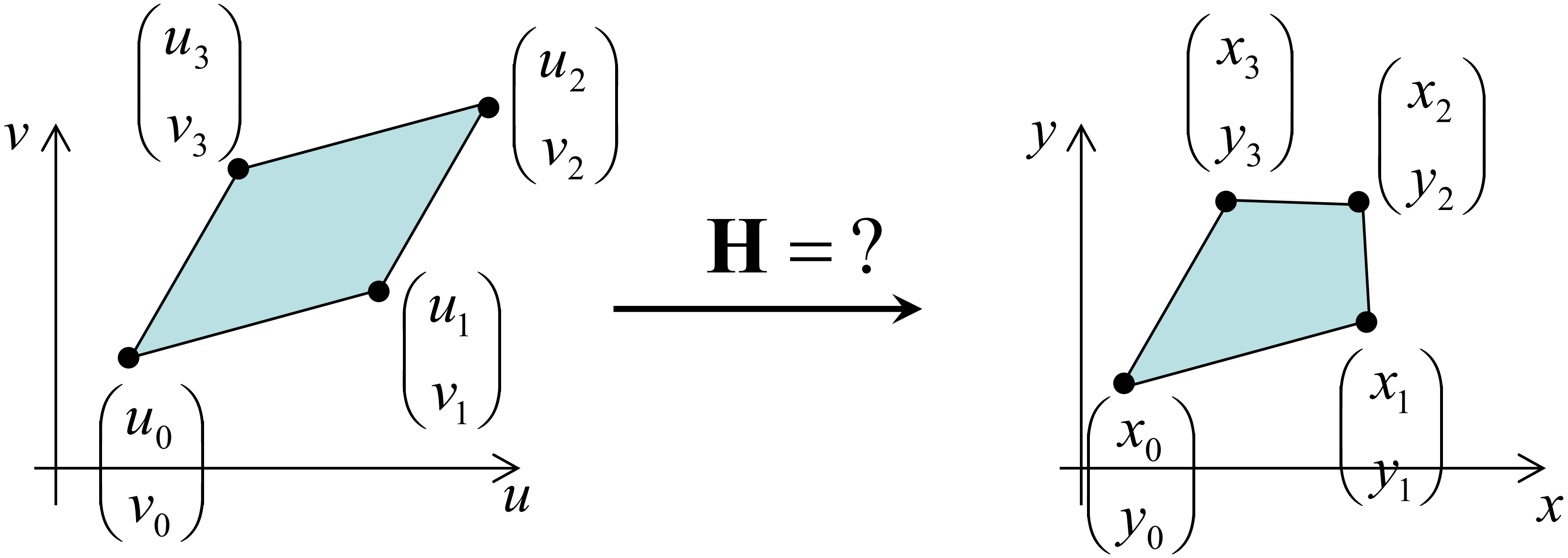


$$\xrightarrow{\mathbf{N} = \mathbf{L}^{-1}\mathbf{T}}$$

$$\mathbf{T} = \begin{bmatrix} 1 & 0 & -u_0 \\ 0 & 1 & -v_0 \\ 0 & 0 & 1 \end{bmatrix}$$

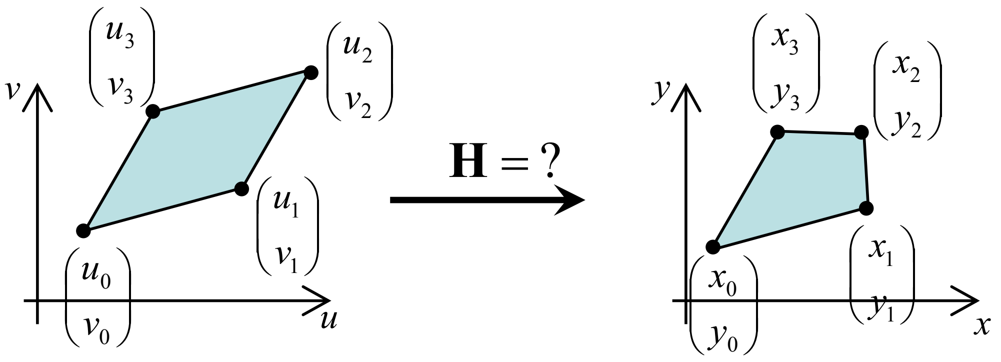
$$\mathbf{L} = \begin{bmatrix} u_1 - u_0 & u_3 - u_0 & 0 \\ v_1 - v_0 & v_3 - v_0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Matriz da Homografia



$$\begin{bmatrix} wx \\ wy \\ w \end{bmatrix} = \begin{bmatrix} h_0 & h_1 & h_2 \\ h_3 & h_4 & h_5 \\ h_6 & h_7 & 1 \end{bmatrix} \begin{bmatrix} u \\ v \\ 1 \end{bmatrix}$$

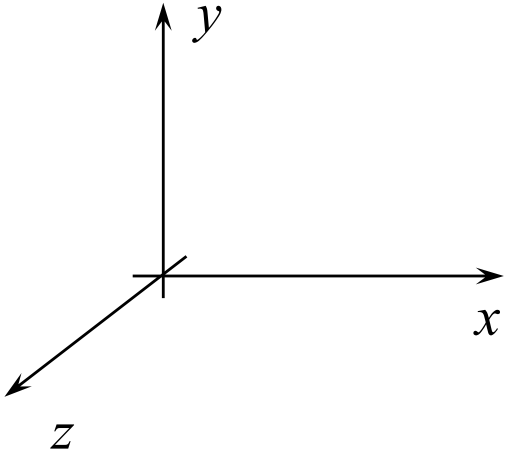
Matriz da Homografia



$$\begin{bmatrix} wx \\ wy \\ w \end{bmatrix} = \begin{bmatrix} h_0 & h_1 & h_2 \\ h_3 & h_4 & h_5 \\ h_6 & h_7 & 1 \end{bmatrix} \begin{bmatrix} u \\ v \\ 1 \end{bmatrix}$$

$$(h_6 u_i + h_7 v_i + 1) x_i = h_0 u_i + h_1 v_i + h_2$$
$$(h_6 u_i + h_7 v_i + 1) y_i = h_3 u_i + h_4 v_i + h_5$$

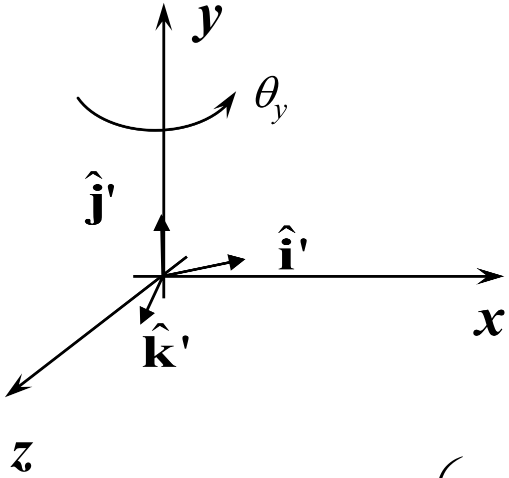
Transformações em 3D (translações e escalas)

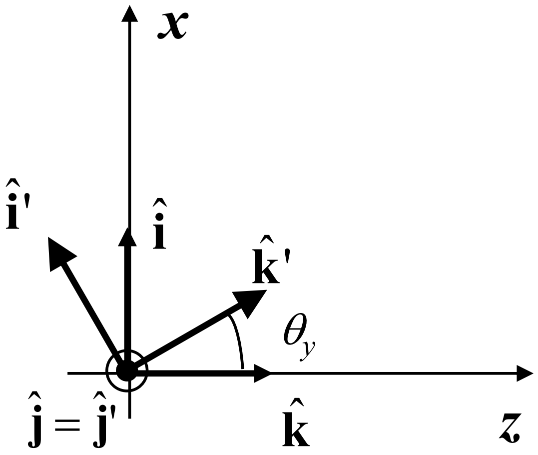


$$\begin{bmatrix} x' \\ y' \\ z' \\ 1 \end{bmatrix} = \left[\begin{array}{ccc|c} 1 & 0 & 0 & t_x \\ 0 & 1 & 0 & t_y \\ 0 & 0 & 1 & t_z \\ \hline 0 & 0 & 0 & 1 \end{array}\right] \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} x' \\ y' \\ z' \\ 1 \end{bmatrix} = \left[\begin{array}{ccc|c} s_x & 0 & 0 & 0 \\ 0 & s_y & 0 & 0 \\ 0 & 0 & s_z & 0 \\ \hline 0 & 0 & 0 & 1 \end{array}\right] \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$

Rotação em torno do eixo *y*



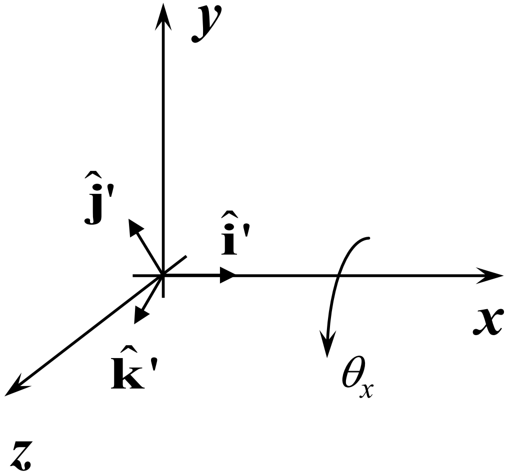


$$\hat{\mathbf{i}}'=\begin{pmatrix}\cos\theta_y\\0\\-\sin\theta_y\end{pmatrix}\qquad \hat{\mathbf{j}}'=\begin{pmatrix}0\\1\\0\end{pmatrix}\qquad \hat{\mathbf{k}}'=\begin{pmatrix}\sin\theta_y\\0\\\cos\theta_y\end{pmatrix}$$

$$\begin{pmatrix}x'\\y'\\z'\end{pmatrix}=\begin{bmatrix}\cos\theta_y & 0 & \sin\theta_y\\0 & 1 & 0\\-\sin\theta_y & 0 & \cos\theta_y\end{bmatrix}\begin{pmatrix}x\\y\\z\end{pmatrix}$$

$$\begin{bmatrix}x'\\y'\\z'\\1\end{bmatrix}=\begin{bmatrix}\cos\theta_y & 0 & \sin\theta_y & 0\\0 & 1 & 0 & 0\\-\sin\theta_y & 0 & \cos\theta_y & 0\\0 & 0 & 0 & 1\end{bmatrix}\begin{bmatrix}x\\y\\z\\1\end{bmatrix}$$

Rotação em torno do eixo *x*

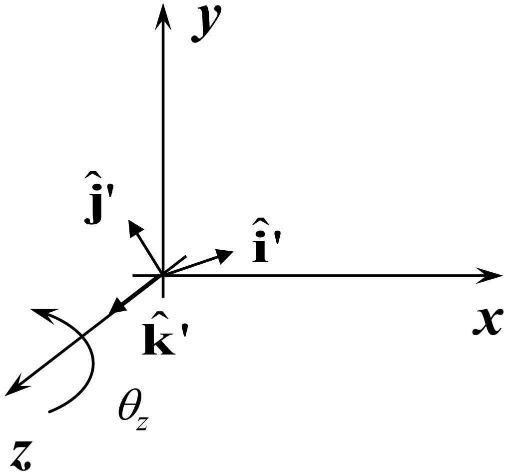


$$\hat{\mathbf{i}}' = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \qquad \hat{\mathbf{j}}' = \begin{pmatrix} 0 \\ \cos\theta_x \\ \sin\theta_x \end{pmatrix} \qquad \hat{\mathbf{k}}' = \begin{pmatrix} 0 \\ -\sin\theta_x \\ \cos\theta_x \end{pmatrix}$$

$$\begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta_x & -\sin\theta_x \\ 0 & \sin\theta_x & \cos\theta_x \end{bmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

$$\begin{bmatrix} x' \\ y' \\ z' \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta_x & -\sin\theta_x & 0 \\ 0 & \sin\theta_x & \cos\theta_x & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$

Rotação em torno do eixo *z*

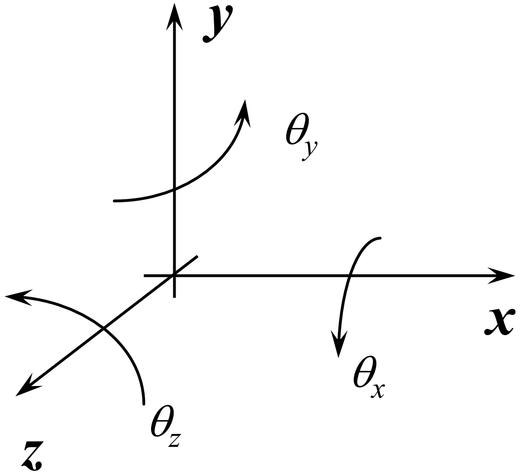


$$\hat{\mathbf{i}}' = \begin{pmatrix} \cos\theta_z \\ \sin\theta_z \\ 0 \end{pmatrix} \quad \hat{\mathbf{j}}' = \begin{pmatrix} -\sin\theta_z \\ \cos\theta_z \\ 0 \end{pmatrix} \quad \hat{\mathbf{k}}' = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

$$\begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} = \begin{bmatrix} \cos\theta_z & -\sin\theta_z & 0 \\ \sin\theta_z & \cos\theta_z & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

$$\begin{bmatrix} x' \\ y' \\ z' \\ 1 \end{bmatrix} = \begin{bmatrix} \cos\theta_z & -\sin\theta_z & 0 & 0 \\ \sin\theta_z & \cos\theta_z & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$

Rotações em torno dos eixos cartesianos

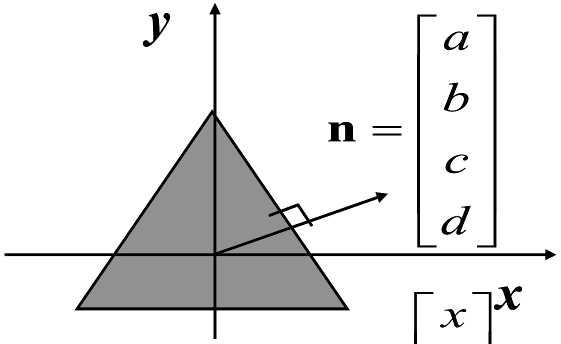


$$\begin{bmatrix} x' \\ y' \\ z' \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta_x & -\sin\theta_x & 0 \\ 0 & \sin\theta_x & \cos\theta_x & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} x' \\ y' \\ z' \\ 1 \end{bmatrix} = \begin{bmatrix} \cos\theta_y & 0 & \sin\theta_y & 0 \\ 0 & 1 & 0 & 0 \\ -\sin\theta_y & 0 & \cos\theta_y & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$

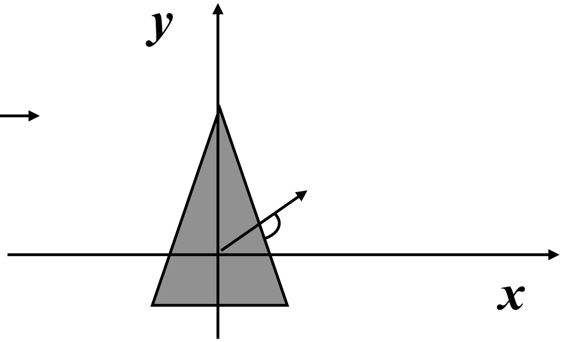
$$\begin{bmatrix} x' \\ y' \\ z' \\ 1 \end{bmatrix} = \begin{bmatrix} \cos\theta_z & -\sin\theta_z & 0 & 0 \\ \sin\theta_z & \cos\theta_z & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$

Transformação de normais



$$\mathbf{n} = \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix}$$

$$\xrightarrow{s_x=0.5}$$



$$\mathbf{p} = \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix} \qquad \mathbf{n} \cdot \mathbf{p} = 0$$

$$\lfloor \mathbf{n}^T \rfloor \{\mathbf{p}\} = \lfloor a \quad b \quad c \quad d \rfloor \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix} = 0$$

$$\lfloor \mathbf{n}^T \rfloor \cdot \{\mathbf{p}\} = \lfloor a \quad b \quad c \quad d \rfloor \mathbf{M}^{-1}\mathbf{M} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix} = 0$$

$$\{\mathbf{p'}\} = \mathbf{M} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$

$$\lfloor \mathbf{n'}^T \rfloor = \lfloor a \quad b \quad c \quad d \rfloor \mathbf{M}^{-1}$$

$$\boxed{\mathbf{n'} = \mathbf{M}^{-T} \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix}}$$

FIM