Pontifícia Universidade Católica
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**Guilherme Coelho Gomes Barros**

# Topology Optimization considering Limit Analysis

**Dissertação de Mestrado**

Dissertation presented to the Programa de Pós–graduação em Engenharia Civil of PUC-Rio in partial fulfillment of the requirements for the degree of Mestre em Engenharia Civil.

Advisor : Prof. Luiz Fernando Martha
Co-Advisor: Prof. Ivan Fábio Mota de Menezes

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To Luiz Eloy Vaz and Jaurès Paulo Feghali

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# Abstract

Barros, Guilherme Coelho Gomes; Martha, Luiz Fernando (Advisor); Menezes, Ivan Fábio Mota (Co-Advisor). **Topology Optimization considering Limit Analysis**. Rio de Janeiro, 2017. 41p. Dissertação de Mestrado – Departamento de Engenharia Civil e Ambiental, Pontifícia Universidade Católica do Rio de Janeiro.

This work presents a full plastic formulation to be applied within topology optimization. The main idea of topology optimization in solid mechanics is to find the material distribution within the domain so that it optimizes a performance measure and satisfies a set of constraints. One might seek to minimize the compliance satisfying that the volume is less than a given value. The aforementioned formulation is the standard topology optimization which has been used widely in literature. Although it provides interesting results, additional requirements must be taken into account when practical application is concerned. Structures are designed considering two main aspects: (i) the structure must not collapse, supporting the applied loads (safety criterion); and (ii) its displacements must be lower than a prescribed bound (serviceability criterion). Consequently, the standard formulation shall be modified, finding the material distribution corresponding to the minimum volume such that the safety criterion is met. Said safety criterion may be defined as restraining the elastic stresses to the yield criterion in the entire domain. This definition has resulted in a new branch in this research field: the stress constrained topology optimization. On the other hand, it is understood that the plastic design criterion is preferable when optimization is intended, since it fully exploits the material strength. Therefore, this work addresses the incorporation of the plastic design criterion into topology optimization as a more advantageous method than standard and stress constrained topology optimization methods. The proposed formulation is an extension of limit analysis, which provides an estimative of the collapse load of a structure directly through mathematical programming, ensuring computational efficiency to the proposed methodology. Lastly, numerical examples are shown to verify plastic topology optimization and the final topology is compared with those provided by standard and stress constrained topology optimization methods.

## Keywords

Plastic Structural Design; Limit Analysis; Topology Optimization.

## Resumo

Barros, Guilherme Coelho Gomes; Martha, Luiz Fernando; Menezes, Ivan Fábio Mota. **Otimização Topológica considerando Análise Limite**. Rio de Janeiro, 2017. 41p. Dissertação de Mestrado – Departamento de Engenharia Civil e Ambiental, Pontifícia Universidade Católica do Rio de Janeiro.

Este trabalho apresenta uma formulação puramente baseada em plasticidade para ser aplicada à otimização topológica. A principal ideia da otimização topológica em mecânica dos sólidos é encontrar a distribuição de material dentro do domínio de forma a otimizar uma medida de performance e satisfazer um conjunto de restrições. Uma possibilidade é minimizar a flexibilidade da estrutura satisfazendo que o volume seja menor do que um determinado valor. Essa é a formulação clássica da otimização topológica, que é vastamente utilizada na literatura. Não obstante fornecer resultados interessantes, condições adicionais devem ser levadas em consideração para viabilizar sua aplicação prática. O projeto estrutural aborda dois aspectos principais: (i) a estrutura não deve colapsar, suportando os carregamentos aplicados (critério de segurança); e (ii) deverá se sujeitar a um valor máximo aceitável de deformação (critério de aceitabilidade). Consequentemente, a otimização topológica clássica deve ser modificada de forma a encontrar a distribuição de material correspondente ao menor volume possível tal que o critério de segurança seja verificado. O referido critério de segurança pode ser definido como limitar as tensões elásticas ao critério de plastificação em todo o domínio. Esta definição resultou em um novo ramo de pesquisa: a otimização topológica com restrições de tensões. Por outro lado, entende-se que o projeto estrutural plástico é preferível quando um projeto ótimo é almejado, uma vez que permite um maior aproveitamento da resistência do material. Dessa forma, este trabalho aborda a incorporação do projeto estrutural plástico à otimização topológica como método mais vantajoso do que a otimização topológica clássica e a com restrições de tensões. A formulação proposta é uma extensão da análise limite, que fornece uma estimativa da carga de colapso de uma estrutura diretamente por meio da programação matemática, assegurando a eficiência computacional da metodologia proposta. De forma a verificar a otimização topológica plástica e comparar a topologia final com as obtidas através da otimização topológica clássica e da com restrição de tensões, são apresentados exemplos numéricos.



## Palavras-chave

Projeto Estrutural Plástico; Análise Limite; Otimização Topológica.

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*"Knowledge is like a sphere, the greater its volume, the larger its contact with the unknown."*

**Blaise Pascal**, *quote.*

# 1
# Introduction

In the field of structural optimization, structural parameters are manipulated in order to render a performance measure its optimal value and satisfy a set of constraints. The performance measure may be a structural response, such as maximum displacement; maximum stress; and structural compliance. It might be a cost-related measure as well, such as the cost itself; the structural weight; and the structural volume. As the performance measure, the constraints might also be a structural response or a cost-related measure. For instance, one may seek the structure with minimum compliance that costs at most a specified value. Another example is the lightest structure that does not undergo plastic behavior under the application of external loads.

Topology optimization is a branch of structural optimization in which the material distribution within a continuum structure is optimized. The standard formulation of topology optimization is to find the material distribution that minimizes the compliance subject to a volume constraint, which is a bounded amount of material to distribute. This formulation provides interesting results and has been widely used in literature. However, regarding practical application, the structural safeness must be verified, among other criteria. In fact, some examples show that the standard topology optimization may not suffice, requiring considerable manual work to determine the final design.

For this reason, a new branch of topology optimization has arisen: the stress constrained topology optimization. Its formulation may be posed as the standard formulation with the addition of stress constraints – i.e. the plastic criterion must be satisfied in the entire domain – or as determining the material distribution which minimizes the volume and meets the stress constraints. This design is said to be an elastic structural design, and may be further improved if a plastic structural design is chosen instead.

Plastic structural design allows the structure to undergo plastic behavior; thus, it provides, in general, a lighter structure. The previous safety criterion is replaced by the following: the structure has to be able to support the applied loads without collapsing. In other words, the elastic design refers to the imminence of yield, while the plastic design deals with the imminence of collapse. The elastic design may be checked by taking the stresses of a linear elastic analysis and confirming that the plastic criterion is met in the entire domain. On the other hand, the plastic design may be verified using an elastoplastic analysis by ascertaining that the collapse load is greater than the applied load.

However, an elastoplastic analysis may be a cumbersome computational task on its own. Thus, to combine it with a structural optimization might make the problem impracticable. Therefore, a direct analysis method for the collapse load ought to be preferred. Such method does exist and it is called limit analysis, which is based in the limit theorems of plasticity, disregarding any elastic structural behavior. This formulation allows the collapse load to be found through mathematical programming. Thereby, a global unique solution is proved to exist and might be found efficiently.

Topology optimization may be modeled through density field, which means the material density is assumed to be a continuous field described by some parameters. Hence, the topology optimization concerns the determination of such parameters. Since the construction of a structure does not allow intermediate values of density, the problem is penalized so that its optimum has value one in certain part of the domain and zero everywhere else.

Given all the above, the main objective of this work is to present a plastic formulation to be applied with topology optimization. The density approach with penalization is used to model the material distribution. Therefore, the optimization problem is solved with a sequential optimization scheme, in which the constraints are retrieved from limit analysis. In order to make the developments clearer representative examples considering both truss model and continuum structures are presented and compared to traditional formulations found in the literature.



## 1.1 Literature review

Topology optimization of continuum structures is a widely covered subject in the literature [1, 2, 3, 4]. Although this technique is a significant step towards a fully automatic structural design, the resulting design may not fulfill design criteria, presenting stress concentrations [5]. To cope with this hurdle, new developments [6] have focused on bringing stress constraints into topology optimization.

However, the solutions available in the literature are, in general, obtained by constraining the elastic stress [5, 6, 7, 8, 9, 10], which may not fully consider structural safety. Therefore, a plastic design should be preferred over an elastic design when attempting to reach an optimal structural design [11, 12, 13].

Nevertheless, it is well known that an elastoplastic analysis is a demanding computational task since it involves the solution of nonlinear systems of equations [14, 15, 16]. Thus, the coupling of an elastoplastic analysis with topology optimization might be impracticable to achieve [17, 18].

The collapse load of a structure may be assessed by limit analysis in spite of elastoplastic analysis. Limit analysis is based on set of fundamental collapse theorems of plasticity [19, 20], and it is applicable for a great variety of problems such as plane stress and plane strain [21, 22, 23, 24]; trusses [25]; frames [26, 27, 28, 29]; bending plates [30, 31, 32]; and three dimensional models [33, 34]. Finite element discretization is adopted in most of these works [35, 36, 37], but it has been shown [38] that the element-free Galerkin approach [39] and the boundary element approach [40, 41] are also possible.

The solution of a limit analysis problem is found by means of an optimization problem [42, 43]. The limit analysis problem had been commonly addressed as a nonlinear programming problem [21, 33, 43, 44] until it was demonstrated [45] that the second-order cone programming is applicable. Second-order cone programming might be viewed as a generalization of the linear programming or a special case of the semidefinite programming [46]. It appeals to limit analysis researchers due to the existence of computationally efficient methods to obtain the solution [47, 48, 49]. Moreover, the solution is proved to be global and independent of initial steps [46, 50]. Therefore, second-order cone programming has been widely adopted on limit analysis in order to provide solutions efficiently and permit the experimentation on more complex numerical examples [31, 51, 52, 53].

The plastic formulation is commonly applied to the topology optimization of truss structures, the so called ground structure optimization [54, 55, 56, 57, 58]. In addition, the elastic formulation for topology optimization of trusses [59] and the plastic formulation considering equal stress limits in compression and tension are equivalent [60]. Although the plastic formulation for topology optimization on truss structures has been widely covered in the literature, this formulation has only recently been explored to continuum structures [61].

## 1.2 Main contributions

This work explores the knowledge on limit analysis to formulate the equilibrium constraints and to deal efficiently with the yield criterion appearing in the plastic topology optimization problem. The work of Kammoun and Smaoui [61] presents a similar formulation. However, the penalization approach adopted in that work has not succeeded in providing good black-and-white results. Therefore, this work presents a new penalization approach to obtain such results within the plastic topology optimization.

## 1.3 Dissertation organization

The remainder of work is divided as follows: Chapter 2 presents the formulation of limit analysis to trusses and continuum structures; Chapter 3 concerns the extension of limit analysis to plastic topology optimization for trusses and continuum structures; Chapter 4 brings verification results of plastic topology optimization and the comparison of final topology with those provided by standard and stress constrained topology optimization methods; and, finally, Chapter 5 consists of conclusions and final remarks regarding this theme.

# 2
# Limit analysis

Limit analysis is a technique to estimate the collapse factor of a structure, which is the maximum scalar that can be multiplied by the applied loads in order to reach structural plastic collapse. It represents an alternative to the elastoplastic analysis. In the elastoplastic analysis a nonlinear system of equilibrium equations must be solved, while in the limit analysis the collapse multiplier is found solving a constrained optimization problem.

Limit analysis is based on a set of fundamental theorems from the theory of plasticity: the lower bound theorem (or static theorem); the upper bound theorem (or kinematic theorem); and the uniqueness theorem. These theorems were firstly formulated by Gvozdev [19] and, independently, by Drucker et al. [20] for a rigid and perfectly plastic model of material structural behavior.

Since elastic behavior is disregarded, the structural behavior may be formulated only in terms of the static variables, for the lower bound theorem, or only in terms of kinematic variables, for the upper bound theorem. Those theorems, as their names suggest, provide bounds to the true collapse load. The uniqueness theorem ensures that if a collapse state is found satisfying both, equilibrium and compatibility, then it associates the actual collapse load which is unique.

In this work the lower bound theorem is adopted in the formulation. This theorem may be stated as: if collapse has not occurred, a safe statically admissible state of stress can be found [20]. In order to base the concept exposed by the lower bound theorem, the definition of safe admissible state of stress must be provided. Consider first a state of stress for which the components $\sigma_{ij}$ are continuous functions of the coordinates. Such a state is called statically admissible if it satisfies the equilibrium conditions:

$$\begin{aligned} \sigma_{ij,j} + b_i &= 0 \\ \sigma_{ij} n_j &= t_i \end{aligned} ; \tag{2-1}$$

in which $b_i$ and $t_i$ are, respectively, the body force and the surface traction in the $i^{\text{th}}$ direction, and $n_j$ is the $j^{\text{th}}$ coordinate of the surface normal vector where traction $t$ is applied. Additionally, the defined admissible state of stress is said to be safe if the yield criterion is verified:

$$f\left(\sigma_{ij}\right) \leq 0\,. \tag{2-2}$$

In summary, the lower bound theorem may be restated as: if an equilibrium distribution of stress can be found which balances the applied load and is everywhere below yield or at yield, the structure will not collapse or will just be at the point of collapse [62]. From this theorem it is possible to enunciate limit analysis as:

$$\lambda^c = \sup\left\{\lambda \in \mathbb{R}^+; \exists\sigma_{ij}, \sigma_{ij,j} + \lambda b_i = 0, \sigma_{ij}n_j = \lambda t_i, f\left(\sigma_{ij}\right) \leq 0\right\}\,, \tag{2-3}$$

in which $\lambda$ is all possible collapse multipliers and $\lambda^c$, the supreme of the set in Eq. (2-3), is the true collapse multiplier. Furthermore, Eq. (2-3) is equivalent to finding the maximum possible collapse multiplier for which a safe statically admissible state of stress can be found.

In order to elucidate the idea of a safe statically admissible state of stress as well as formulate limit analysis as a mathematical programming problem, first truss structures are addressed. Afterwards, continuum structures in plane stress are formulated by means of the finite element method.



## 2.1 Truss structures

The truss structure is addressed in order to clarify the concept of safe statically admissible state of stress. Particularly, for truss structures, a state of stress is a collection of axial forces within each member. Hence, the set of states of stress is defined as:

$$N = \{\mathbf{N} \in \mathbb{R}^m\} = \mathbb{R}^m\,, \tag{2-4}$$

in which $m$ is the number of members of a truss structure. From this set it may be defined a subset of the statically admissible states of stress as:

$$\bar{S} = \left\{\mathbf{N} \in N; \sum_{j=1}^{m} a_{ij}N_j = F_i \ \forall i \in \{1, 2, \cdots, d_f\}\right\}\,, \tag{2-5}$$

in which $n$ is the number of joints (nodes) and $d_f$ is the number of degrees of freedom of a truss structure. Moreover, in Eq. (2-5), the term $a_{ij}$ represents the contribution of member $j$ into the equilibrium equation of the $i^{\text{th}}$ degree of freedom, onto which it is applied the external force $F_i$. It is worth mentioning that, for truss structures, it is verified

$$d_f = 2n - s\,, \tag{2-6}$$

in which $s$ is the number of supports applied to a truss structure. Additionally, a state of stress of a truss is said to be safe if the yield criterion is verified everywhere within the structural domain. Hence, the set of safe stress states is determined as:

$$\hat{S} = \left\{ \mathbf{N} \in N; N_j^l \leq N_j \leq N_j^u \ \forall j \in \{1, 2, \cdots, m\} \right\} , \tag{2-7}$$

in which $N_j^l$ and $N_j^u$ are, respectively, the lower bound and the upper bound for the internal force of the $j^{\mathrm{th}}$ member of a truss structure. From the formal definition of safe stress states and statically admissible stress states it is possible to define the set of safe statically admissible stress states as:

$$S = \hat{S} \cap \bar{S} . \tag{2-8}$$

In cases of statically determined structures, there is only one possible statically admissible state of stress. Therefore, it is easy to determine whether this state of stress is safe or not. For the statically indeterminate truss of Figure 2.1, any stress vector of size five consists of a state of stress. In addition, this structure has the following set of equilibrium equations:

$$\begin{aligned} \sum F_x^3 &= -\frac{4}{5}N_3 - N_4 - \lambda P = 0 \\ \sum F_y^3 &= N_2 + \frac{3}{5}N_3 = 0 \\ \sum F_x^4 &= \frac{4}{5}N_1 + N_4 = 0 \\ \sum F_y^4 &= \frac{3}{5}N_1 + N_5 = 0 \end{aligned} , \tag{2-9}$$

in which the applied load $P$ is multiplied by the scalar $\lambda$. Eq. (2-9) can be written in matrix form as:

$$[A]\{N\} = \lambda\{F\} , \tag{2-10}$$

in which

$$[A] = \begin{bmatrix} 0 & 0 & -\frac{4}{5} & -1 & 0 \\ 0 & 1 & \frac{3}{5} & 0 & 0 \\ \frac{4}{5} & 0 & 0 & 1 & 0 \\ \frac{3}{5} & 0 & 0 & 0 & 1 \end{bmatrix} \tag{2-11}$$

is the equilibrium matrix and

$$\{F\} = \begin{Bmatrix} P \\ 0 \\ 0 \\ 0 \end{Bmatrix} \tag{2-12}$$

is the vector of applied loads.

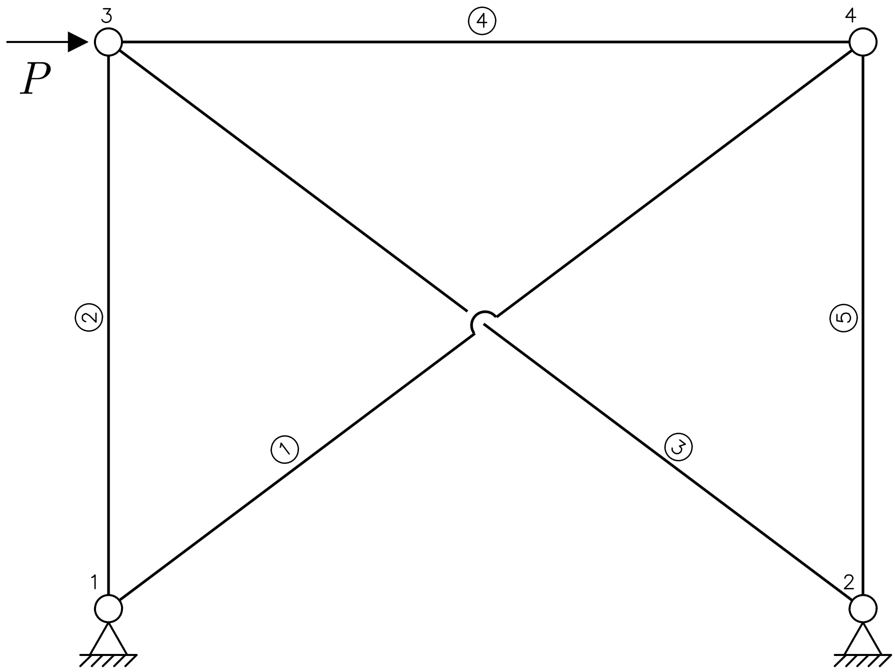


Figure 2.1: Statically indeterminate truss



It is easy to see, for the structural model shown in Figure 2.1, that the set of equilibrium equation are linearly independent, thus matrix $[A]$ has full row rank equal to four. Therefore, the set of statically admissible states of stress is a subset of $\mathbb{R}^5$ with dimension one. Choosing $N_1$ as independent parameter, the other components of $\mathbf{N}$ are written as:

$$\begin{aligned} N_2 &= -\frac{3}{5}N_1 + \lambda\frac{3}{4}P \\ N_3 &= N_1 - \lambda\frac{5}{4}P \\ N_4 &= -\frac{4}{5}N_1 \\ N_5 &= -\frac{3}{5}N_1 \end{aligned} \quad . \tag{2-13}$$

Assuming $N_i^l = -N_\mathrm{y}$ and $N_i^u = N_\mathrm{y}$ in the example of Figure 2.1, the set of safe statically admissible states stress is written as:

$$S = \left\{ N_1 \in \mathbb{R};\ \begin{array}{c} -N_\mathrm{y} \le N_1 \le N_\mathrm{y} \\ -N_\mathrm{y} \le -\frac{3}{5}N_1 + \lambda\frac{3}{4}P \le N_\mathrm{y} \\ -N_\mathrm{y} \le N_1 - \lambda\frac{5}{4}P \le N_\mathrm{y} \\ -N_\mathrm{y} \le -\frac{4}{5}N_1 \le N_\mathrm{y} \\ -N_\mathrm{y} \le -\frac{3}{5}N_1 \le N_\mathrm{y} \end{array} \right\} . \tag{2-14}$$

This set might be represented by means of auxiliary variables:

$$x = \frac{\lambda P}{N_\mathrm{y}} , \tag{2-15}$$

$$y = \frac{N_1}{N_\mathrm{y}}\,. \tag{2-16}$$

It is shown in Figure 2.2 the set of safe statically admissible states of stress, represented by the gray region. Every point in $x-y$ space is a statically admissible stress state. Additionally, each line color is related to a member; solid lines represent the limit of the yield criterion due to tension; while dashed lines represent the limit of the yield criterion under compression. Each member yield criterion defines a feasible region and the intersection of all feasible regions gives the set of safe statically admissible states of stress.

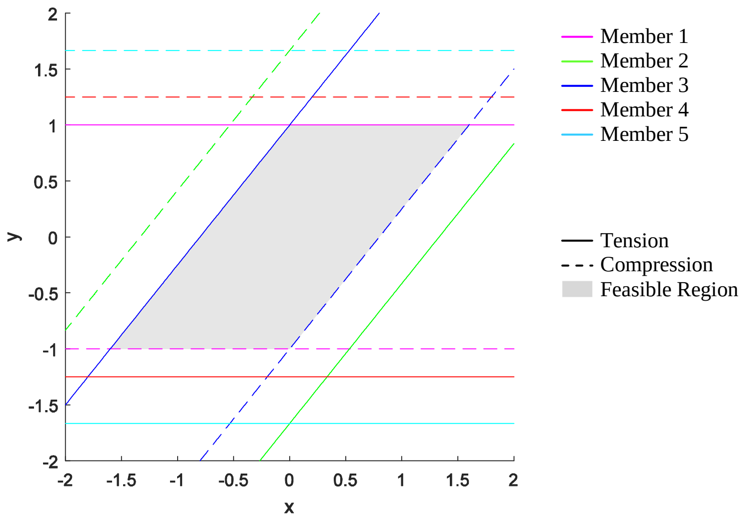


Figure 2.2: Feasible region

As previously stated, the collapse multiplier $\lambda^c$ corresponds to the maximum value of $\lambda$. From Figure 2.2 it can be observed that the maximum possible value of variable $x$ is:

$$x = \frac{\lambda P}{N_\mathrm{y}} = 1.6\,, \tag{2-17}$$

from which the true collapse multiplier is:

$$\lambda^c = 1.6\frac{N_\mathrm{y}}{P}\,. \tag{2-18}$$

Thereby, it is possible to find the collapse load of the structure only through its equilibrium equations and yield criterion, without the need to carry out a tiresome elastoplastic analysis. Even though the procedure was carried out graphically, it may be generalized to any truss structural model by means of linear programming as:

$$\begin{aligned} \max_{\lambda,\{N\}} \quad & \lambda \\ \text{s.t.} \quad & \begin{cases} [A]\,\{N\} = \lambda\,\{F\} \\ \left\{N^l\right\} \leq \{N\} \leq \{N^u\} \end{cases} . \end{aligned} \tag{2-19}$$

## 2.2
## Continuum structures

For the continuum case, the collapse load might be found from Eq. (2-3). However, it would require the finding of a function for each stress component satisfying equilibrium and yield criterion for each value of $\lambda$. Therefore, the finite element method (FEM) is employed in order to obtain a discrete formulation solvable by mathematical programming.

The equilibrium equation may be formulated from the principle of virtual displacement:

$$\int_{\Omega} \{\delta\dot{\varepsilon}\}^{\mathrm{T}} \{\sigma\}\, d\Omega = \left\{\delta\dot{D}\right\}^{\mathrm{T}} \{F\} \;, \tag{2-20}$$

in which $\{\sigma\} = \left\{\sigma_{xx} \quad \sigma_{yy} \quad \tau_{xy}\right\}^{\mathrm{T}}$ is the stress vector; $\{\delta\dot{\varepsilon}\}$ is the vector of corresponding virtual strain rate; $\{F\}$ is the vector of applied loads; and $\left\{\delta\dot{D}\right\}$ is the nodal virtual velocity vector. Through the finite element discretization, instead of continuum functions, piecewise continuum functions are adopted for each stress component. Therefore, Eq. (2-20) is applied for each element:

$$\int_{\Omega^e} \{\delta\dot{\varepsilon}^e\}^{\mathrm{T}} \{\sigma^e\}\, d\Omega^e = \left\{\delta\dot{d}^e\right\}^{\mathrm{T}} \{f^e\} \;, \tag{2-21}$$

in which $\left\{\delta\dot{d}^e\right\}$ and $\{f^e\}$ are the nodal virtual velocity vector and the load vector of element $e$, respectively. The time derivative of the compatibility equation commonly used in the FEM leads to:

$$\{\delta\dot{\varepsilon}^e\} = [B^e] \left\{\delta\dot{d}^e\right\} \;. \tag{2-22}$$

The difference between the elastic formulation and the plastic one is that in the plastic formulation an independent interpolation of the stress field is adopted. This interpolation is given as:

$$\{\sigma^e\} = [H^e] \{\beta^e\} \;, \tag{2-23}$$

in which $[H^e]$ is the element stress interpolation matrix and $\{\beta^e\}$ is the element stress parameter vector.

Substituting Eq. (2-22) and Eq. (2-23) into Eq. (2-20) one might find

$$\left\{\delta\dot{d}^e\right\}^{\mathrm{T}} \left(\int_{\Omega^e} [B^e]^{\mathrm{T}} [H^e]\, d\Omega^e\right) \{\beta^e\} = \left\{\delta\dot{d}^e\right\}^{\mathrm{T}} \{f^e\} \;. \tag{2-24}$$

Since the virtual velocity vector is arbitrary, it must hold:

$$\left(\int_{\Omega^e} [B^e]^{\mathrm{T}} [H^e]\, d\Omega^e\right) \{\beta^e\} = \{f^e\} \;, \tag{2-25}$$

or,

$$[A^e]\{\beta^e\} = \{f^e\} \,, \tag{2-26}$$

with

$$[A^e] = \int\limits_{\Omega^e} [B^e]^{\mathrm{T}} [H^e] \, d\Omega^e \,. \tag{2-27}$$

Adding up the contribution of each element, it is possible to assemble a global equilibrium equation, as:

$$[A]\{\beta\} = \{F\} \,. \tag{2-28}$$

Thereby, the limit analysis may be formulated as:

$$\begin{aligned} \max_{\lambda,\{\beta\}} \quad & \lambda \\ \text{s.t.} \quad & \begin{cases} [A]\{\beta\} = \lambda\{F\} \\ f^i\left(\sigma\left(\{\beta\}\right)\right) \leq 0 \;\; i = 1, \cdots, q \end{cases} \end{aligned} \,, \tag{2-29}$$

in which $f^i\left(\{\sigma\}\right) = f\left(\begin{Bmatrix}\sigma^i_{xx} & \sigma^i_{yy} & \tau^i_{xy}\end{Bmatrix}^{\mathrm{T}}\right)$ is the constraint associated to the yield criterion at the $i$th verification point and $q$ is the number of points where the yield criterion must be verified.

While the velocity interpolation within an element is a well-covered subject among researchers in this field, the stress interpolation may not be as well-known. Therefore, the next section of this paper will further illustrate this matter.

### 2.2.1
### Element formulation

Possibly, the most straightforward element formulation is to employ linear velocity field and constant stress field over a triangular shaped element. In this case, the $[B]$ matrix is constant and the stress parameters are the stress itself, so

$$[A^e] = t\, a\, [B^e]^{\mathrm{T}} \,, \tag{2-30}$$

in which $t$ is the thickness of the element and $a$ is its area.

Alternatively, a linear interpolation of the stress field may be used. In this case,

$$\begin{Bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \tau_{xy} \end{Bmatrix} = \begin{bmatrix} \varphi_1 & 0 & 0 & \varphi_2 & 0 & 0 & \varphi_3 & 0 & 0 \\ 0 & \varphi_1 & 0 & 0 & \varphi_2 & 0 & 0 & \varphi_3 & 0 \\ 0 & 0 & \varphi_1 & 0 & 0 & \varphi_2 & 0 & 0 & \varphi_3 \end{bmatrix} \begin{Bmatrix} \sigma_{xx}^1 \\ \sigma_{yy}^1 \\ \tau_{xy}^1 \\ \sigma_{xx}^2 \\ \sigma_{yy}^2 \\ \tau_{xy}^2 \\ \sigma_{xx}^3 \\ \sigma_{yy}^3 \\ \tau_{xy}^3 \end{Bmatrix}, \quad (2\text{-}31)$$

in which $\varphi_i$ is the shape function corresponding to the $i^{\text{th}}$ node of the element, the same used in the velocity interpolation [35, 36]; and, for instance, $\sigma_{xx}^i$ is $\sigma_{xx}$ at the element $i^{\text{th}}$ node.

Instead of those listed above, a linear interpolation based on the Airy stress function is used here, *i.e.*:

$$\begin{aligned} \sigma_{xx} &= \frac{\partial^2 \phi}{\partial \eta^2} \\ \sigma_{yy} &= \frac{\partial^2 \phi}{\partial \xi^2} \\ \tau_{xy} &= -\frac{\partial^2 \phi}{\partial \xi \partial \eta} \end{aligned} \quad , \quad (2\text{-}32)$$

in which

$$\phi\left(\xi,\eta\right) = \frac{\beta_1}{2}\xi^2 + \beta_2 \xi\eta + \frac{\beta_3}{2}\eta^2 + \frac{\beta_4}{6}\xi^3 + \frac{\beta_5}{2}\xi^2\eta + \frac{\beta_6}{2}\xi\eta^2 + \frac{\beta_7}{6}\eta^3 \,. \quad (2\text{-}33)$$

Consequently, Eq. (2-33) is rewritten as

$$\{\sigma\} = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & \xi & \eta \\ 1 & 0 & 0 & \xi & \eta & 0 & 0 \\ 0 & -1 & 0 & 0 & -\xi & -\eta & 0 \end{bmatrix} \begin{Bmatrix} \beta_1 \\ \beta_2 \\ \beta_3 \\ \beta_4 \\ \beta_5 \\ \beta_6 \\ \beta_7 \end{Bmatrix}, \quad (2\text{-}34)$$

and Eq. (2-29) may be restated as

$$\begin{aligned} \max_{\lambda,\{\beta\},\{\sigma\}} \quad & \lambda \\ \text{s.t.} \quad & \begin{cases} [A]\{\beta\} = \lambda\{F\} \\ \{\sigma\} = [D]\{\beta\} \\ f^i\left(\{\sigma\}\right) \leq 0 \;\; i = 1,\cdots,q \end{cases} \end{aligned} \quad , \quad (2\text{-}35)$$

in which matrix $[D]$ is an assemblage of the evaluation of $[H^e]$ to the element nodes.

Since the principle of virtual displacements, as per Eq. (2-20), is the weak form of the differential equilibrium equation, the Airy stress function approach is more advantageous because it intrinsically verifies the differential equilibrium equation in the parametric space.

Another crucial aspect of limit analysis is how to deal with the yield criterion effectively. This issue will be addressed in Subsection 2.2.2.

### 2.2.2
### Treatment of plastic constraints

If no proper care is taken when formulating the mathematical programming problem, which arises from limit analysis, it may result in a difficult and time-consuming nonlinear problem. As previously demonstrated, since the problem is assumed to be initially rigid, the equilibrium equation is a set of linear equality constraints. Consequently, the only source of nonlinearity lies in the yield criterion.



The most common yield criteria used in practice are suitable to be shaped into semidefinite conic form, while in some particular cases the criteria may be shaped into second-order cone constraints. It is presented in [63] the conic representation of several yield criteria found in the literature. Particularly, the Mohr-Coulomb, Rankine and Tresca criteria are representable as positive semidefinite cones, while the von Mises and Drucker Prager criteria may be written as second-order conic constraints as firstly shown in the work of Andersen et al. [45].

A second-order cone constraint is, given a $n$ dimensional vector $\{x\}$,

$$\|\{x_2, \ldots, x_n\}\| \leq x_1 \, , \tag{2-36}$$

or, more literally, the first entry of a vector must be greater than or equal to the norm of the remaining vector. Eq. (2-36) represents a $n$ dimensional quadratic cone along $x_1$.

The von Mises criterion, for instance, may be viewed as a second-order cone constraint with some algebraic manipulation. Initially, the von Mises criterion may be stated as:

$$f\left(\{\sigma\}\right) = \sqrt{\sigma_{xx}^2 + \sigma_{xx}^2 - \sigma_{xx}\sigma_{yy} + 3\tau_{xy}^2} - \sigma_{\mathrm{y}} \, , \tag{2-37}$$

in which $\sigma_{\mathrm{y}}$ is the material yield stress. Alternatively, Eq. (2-37) could be expressed in matrix form as:

$$f\left(\{\sigma\}\right) = \sqrt{\{\sigma\}^{\mathrm{T}} [M] \{\sigma\}} - \sigma_{\mathrm{y}} \, , \tag{2-38}$$

in which

$$[M] = \begin{bmatrix} 1 & -\frac{1}{2} & 0 \\ -\frac{1}{2} & 1 & 0 \\ 0 & 0 & 3 \end{bmatrix} . \tag{2-39}$$

Since $[M]$ is a positive definite matrix, it is possible to find its Cholesky decomposition $[M] = [L]^{\mathrm{T}}[L]$; thus, Eq. (2-38) may be rewritten as:

$$f(\{\sigma\}) = \sqrt{\{\sigma\}^{\mathrm{T}}[L]^{\mathrm{T}}[L]\{\sigma\}} - \sigma_{\mathrm{y}} . \tag{2-40}$$

Defining

$$\{y\} = [L]\{\sigma\} , \tag{2-41}$$

Eq. (2-40) is redefined as:

$$f(\{\sigma\}) = \sqrt{\{y\}^{\mathrm{T}}\{y\}} - \sigma_{\mathrm{y}} = \|\{y\}\| - \sigma_{\mathrm{y}} . \tag{2-42}$$

Using Eq. (2-40), it is possible to rewrite Eq. (2-29) as:

$$\begin{array}{ll} \max\limits_{\lambda,\{\beta\},\{\sigma\},\{y\}} & \lambda \\ \text{s.t.} & \begin{cases} [A]\{\beta\} = \lambda\{F\} \\ \{\sigma\} = [D]\{\beta\} \\ \{y\} = [L]\{\sigma\} \\ \|\{y^i\}\| \leq \sigma_{\mathrm{y}}^i \ \forall i = 1, \cdots, q \end{cases} \end{array} , \tag{2-43}$$

which is a second-order cone programming problem.

# 3
# Topology optimization considering limit analysis

The plastic topology optimization presented in this work is based on the lower bound limit theorem of plasticity. The idea is to minimize the volume of the structure while it is still possible to find a statically admissible state of stress. In this formulation, the relationship between the structural response and the density field is given by the yield criterion.

The truss structure case is addressed first for elucidation purpose. In this case, the continuum density design, in which the density is allowed to be any value between zero and one, may be formulated as:

$$\begin{aligned} \min_{\{\rho\},\{N\}} \quad & \sum_{i=1}^{m} a_i L_i \rho_i \\ \text{s.t.} \quad & \begin{cases} [A]\{N\} = \{F\} \\ \rho_i N_i^l \leq N_i \leq \rho_i N_i^u \;\; i = 1, \cdots, m \\ 0 \leq \rho_i \leq 1 \;\; i = 1, \cdots, m \end{cases} \end{aligned}, \tag{3-1}$$

in which $a_i$, $L_i$ and $\rho_i$ are the cross section area, the length and the density of the $i^{\text{th}}$ element, respectively.

A similar approach, the so called ground structure, would be to optimize the cross section area of each element instead of the density. The density approach is adopted herein because of its similarity with the formulation for continuum media. Although plastic topology optimization for truss structures has been widely explored in the literature, little was done in applying this formulation to continuum media.

However, from the limit analysis formulation and the knowledge on plastic topology optimization of truss structures, it should be quite straightforward to formulate the plastic topology optimization of continuum structures using finite element technique. This formulation is posed as:

$$\begin{aligned} \min_{\{\rho\},\{\beta\},\{\sigma\},\{y\}} \quad & \sum_{i=1}^{m} t \rho_i a_i \\ \text{s.t.} \quad & \begin{cases} [A]\{\beta\} = \lambda\{F\} \\ \{\sigma\} = [D]\{\beta\} \\ \{y\} = [L]\{\sigma\} \\ \left\| \left\{ y^i \right\} \right\| \leq \rho_i \sigma_{\text{y}}^i \;\; \forall i = 1, \cdots, q \\ 0 \leq \rho_i \leq 1 \end{cases} \end{aligned}. \tag{3-2}$$

Obviously, the formulation in Eq. (3-2) allows the density of each element to be any real value between zero and one. Nevertheless, a possible-to-construct design would have only zero or one density values. This may be achieved by using the solid isotropic material with penalization (SIMP) approach. The idea of the SIMP is to penalize the continuous density field in order to make it tend to zero or one.

In the work of Kammoun and Smaoui [61] the penalization is applied to the objective function as:

$$\begin{array}{ll} \min\limits_{\{\rho\},\{\beta\},\{\sigma\},\{x\}} & \sum\limits_{i=1}^{m} t\rho_i^{\frac{1}{p}} a_i \\ \text{s.t.} & \begin{cases} [A]\{\beta\} = \lambda\{F\} \\ \{\sigma\} = [D]\{\beta\} \\ \{y\} = [L]\{\sigma\} \\ \left\|\left\{y^i\right\}\right\| \leq \rho_i \sigma_{\mathrm{y}}^i \ \forall i = 1, \cdots, q \\ 0 \leq \rho_i \leq 1 \end{cases} \end{array}, \tag{3-3}$$

in which $p$ is the penalization parameter. In the case of $p = 1$, the solution is the continuum density field. However, as $p$ becomes greater the elements with densities $0 < \rho < 1$ becomes heavier and shall be removed. Notwithstanding, this approach has two main disadvantages: (i) if any density approaches to zero the gradient at the corresponding entry tends to infinity; and (ii) since the penalization occurs within a summation it loses effectiveness componentwise, resulting in a topology with lots of gray region.

In order to overcome these hurdles, it is proposed in this work an innovative penalization approach. This approach consists of penalizing intermediate densities at the yield criterion:

$$\begin{array}{ll} \min\limits_{\{\rho\},\{\beta\},\{\sigma\},\{x\}} & \sum\limits_{i=1}^{m} t\rho_i a_i \\ \text{s.t.} & \begin{cases} [A]\{\beta\} = \lambda\{F\} \\ \{\sigma\} = [D]\{\beta\} \\ \{y\} = [L]\{\sigma\} \\ \left\|\left\{y^i\right\}\right\| \leq \rho_i^p \sigma_{\mathrm{y}}^i \ \forall i = 1, \cdots, q \\ 0 \leq \rho_i \leq 1 \end{cases} \end{array}. \tag{3-4}$$

Since

$$0 \leq \rho \leq 1, p \geq 1 \Rightarrow \rho^p \leq \rho\,, \tag{3-5}$$

the yield limit of intermediate value densities are further reduced; thus, these densities are forced towards zero or one. From Eq. (3-4) it can be seen that each density is penalized separately and the derivative of the penalized function is well-defined at zero.

# 4
# Numerical results

In this section, numerical examples are presented to verify plastic topology optimization and the final topology is compared with those provided by standard and stress constrained topology optimization methods.

While running the examples shown below, it was observed that concentrated loads may cause a local plastification in the neighbor boundaries of the load, leading to a localized collapse mechanism. Therefore, the resulting collapse load would not reflect reality. The same drawback occurs in stress constrained topology optimization, since concentrated load often implies stress concentration [5]. A possible alternative around this problem is to distribute the load over a small portion of the boundary near the application point.

## 4.1
## MBB-Beam

The MBB-beam is a benchmark example in topology optimization. It consists of a simply supported beam with a mid-span applied load. It is shown in Figure 4.1 the model, considering symmetry, used in the analysis. The adopted material follows the von Mises criterion and has yield limit of 350 MPa; the applied load is 1,500 N; the plate thickness is 1 mm; and the geometry of the plate is given by $L = 100$ mm. The stress constrained topology optimization performed by Holmberg, E. et al [5] considered the material Young's modulus 71 GPa and Poisson's ratio 0.33.

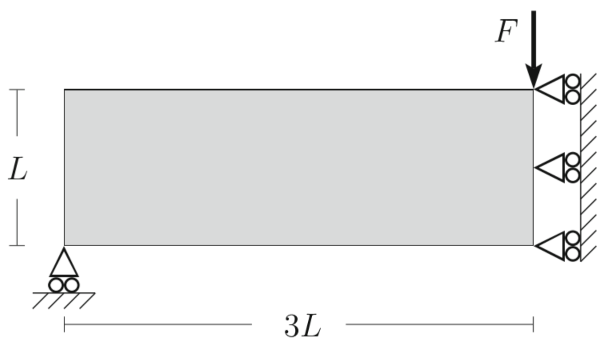


Figure 4.1: Geometry of the MBB-Beam problem [5]

The result obtained by Talischi et al [64] with the standard formulation of topology optimization is shown in Figure 4.2. In the work of Holmberg et al [5] on stress constrained optimization, a main disadvantage is made clear by the results: several different final results are achieved for the same problem depending on setup parameters. One result obtained in that work is presented in Figure 4.3.

In order to avoid the aforementioned localized collapse mechanism problem, the load is distributed over two adjacent nodes on the top boundary. Since the support on the bottom boundary is isolated, its reaction will also cause the localized collapse mechanism problem. Hence, two more supports are added to adjacent nodes at the bottom boundary. The resulting topology is shown in Figure 4.4. Figure 4.5 illustrates the map of stresses of the resulting topology upon collapse. The mesh used to obtain these results have 1,200 elements.

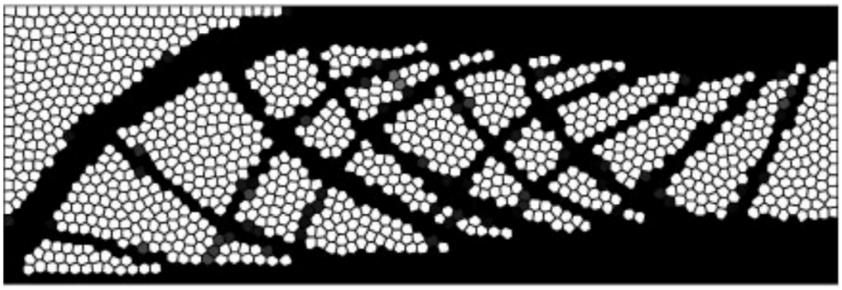

Figure 4.2: MBB-Beam by standard topology optimization

Figure 4.3: MBB-Beam by stress constrained topology optimization

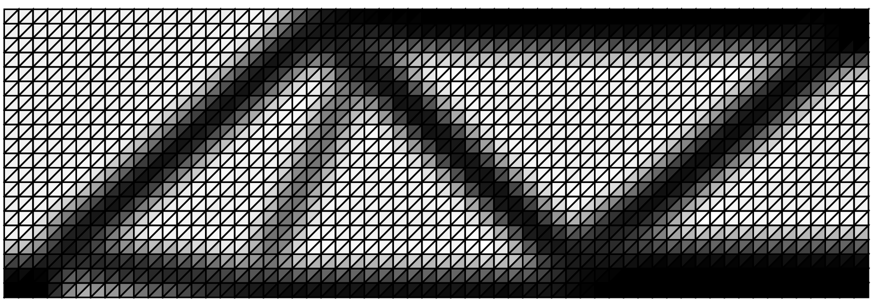

Figure 4.4: MBB-Beam by plastic topology optimization

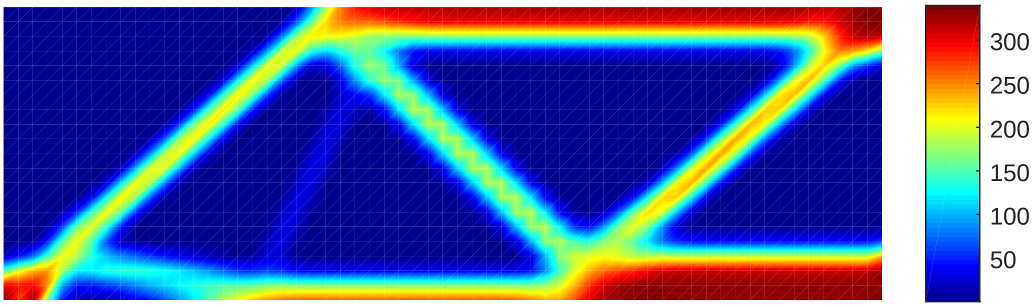


Figure 4.5: Map of stresses of resulting topology

## 4.2 L-shaped beam

The L-shaped beam is a popular example for stress constrained topology optimization [5, 6]. It owes its popularity to the stress concentration at the corner. The structural model adopted within the optimization is presented in Figure 4.6. The employed material is the same of the MBB-Beam problem. The geometry is given by $L = 200$ mm and thickness equal to 1 mm. A 1,500 N point load is applied to the structure.

Using PolyTop [65], the standard topology optimization is performed in this structure. The resulting topology is presented in Figure 4.7. On the other hand, the result of stress constrained topology optimization [5] is presented in Figure 4.8. It is presented in Figure 4.9 the optimum topology provided by plastic topology optimization and, in Figure 4.10, the stresses at plastic collapse of this optimum design.

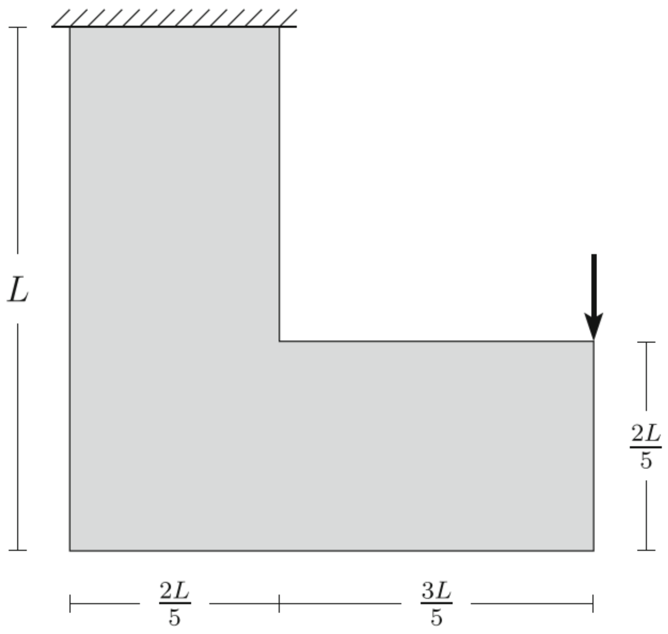


Figure 4.6: Geometry of the L-shaped beam problem [5]



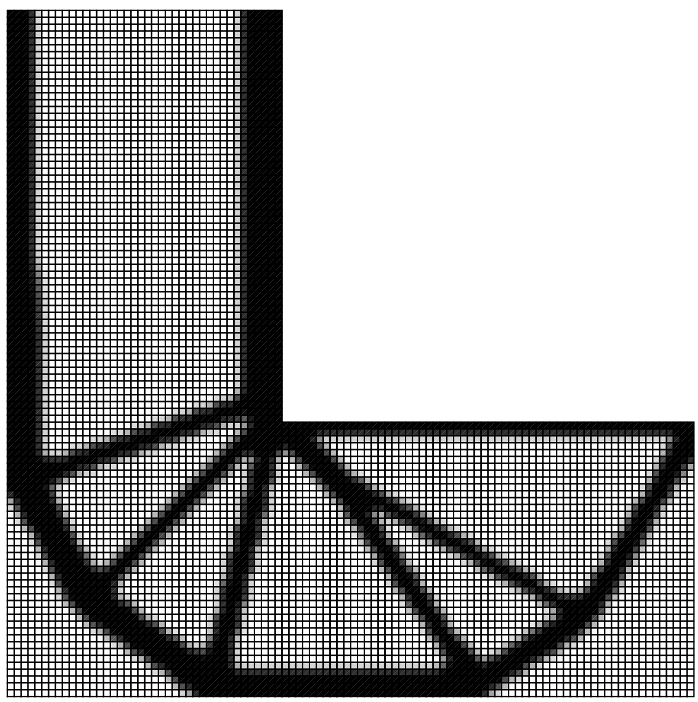

Figure 4.7: L-shaped beam with standard topology optmization

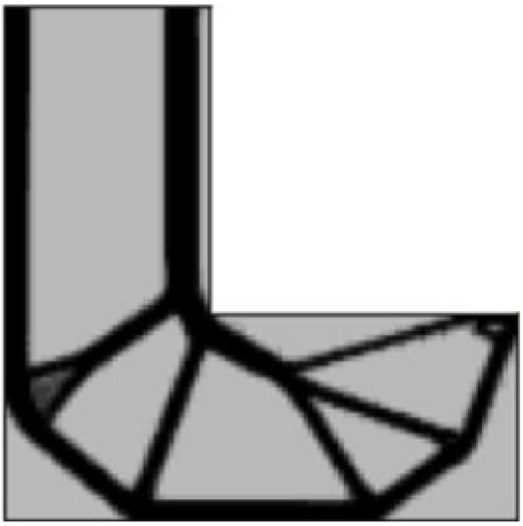

Figure 4.8: L-shaped beam with stress constrained topology optimization [5]

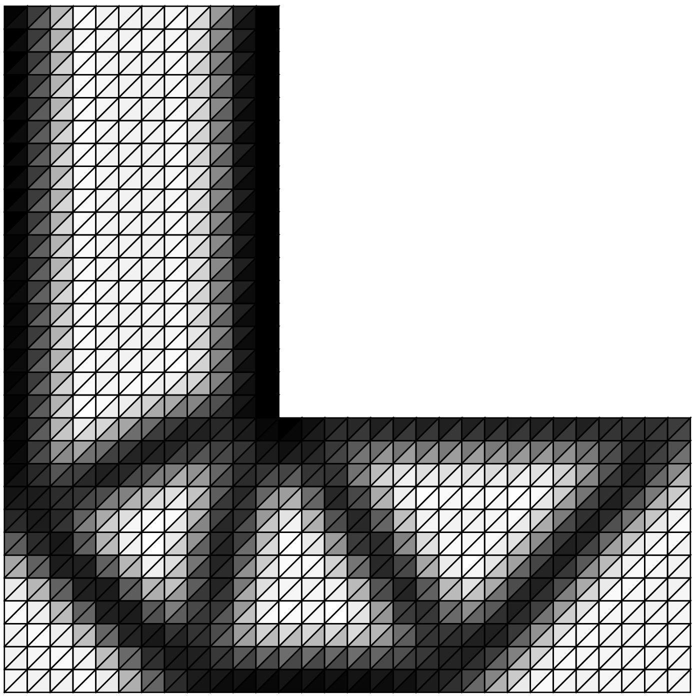

Figure 4.9: L-shaped beam with plastic topology optimization

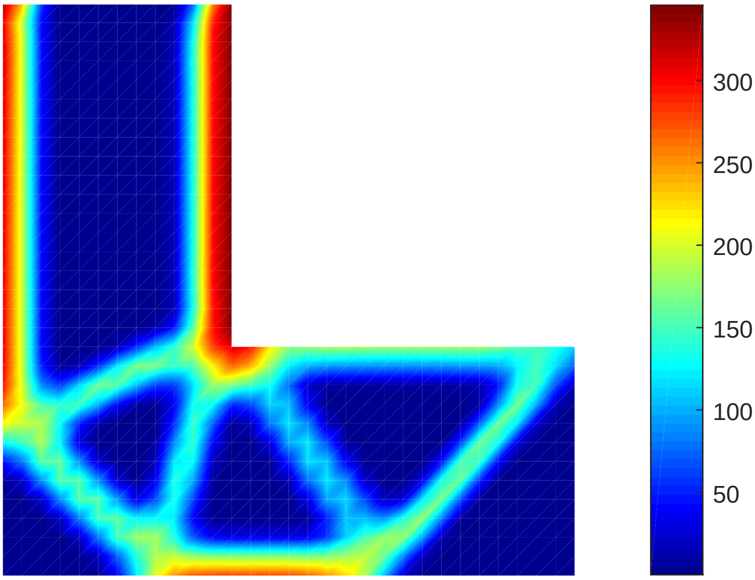


Figure 4.10: Stresses map of the optimal topology through plastic topology optimization



## 4.3 Long cantilever beam

The long cantilever beam problem, shown in Figure 4.11, is addressed in the work of Kammoun and Smaoui [61] and, therefore, it is considered herein for comparison purposes. The structural domain has $H$ = 1 m and $L$ =4 m; the material adopted follows the von Mises yield criterion with yield stress $\sigma_{\mathrm{y}}$ = 1kPa; and a 0.1 kN load is applied. In order to avoid localized collapse mechanism, the load is distributed over a portion of the boundary $b$ = 10cm.

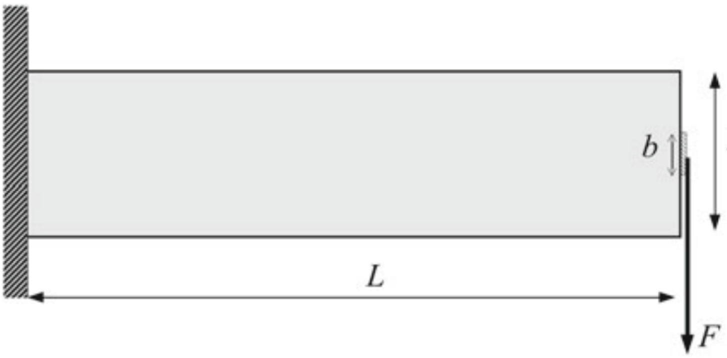


Figure 4.11: Long cantilever beam

The result of standard topology optimization for this problem found by [66] is shown in 4.12. The result found by Kammoun Z. and Smaoui H. [61] is represented in 4.13. And the resulting topology using the presented formulation is illustrated in 4.14. The collapse state of stress of the optimized structure is shown in 4.15.

Figure 4.12: Standard topology optimization for the long cantilever beam [66]

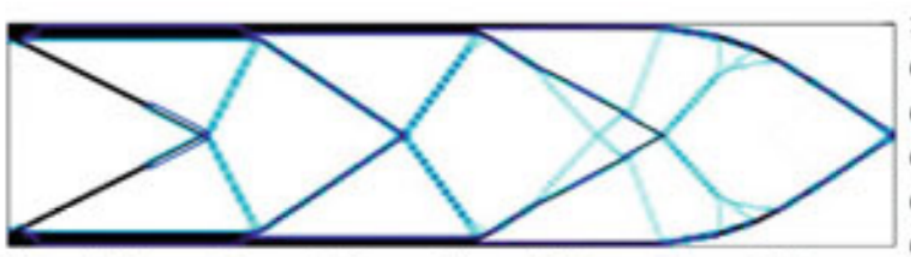

Figure 4.13: Plastic topology optimization with penalization in objective function [61]

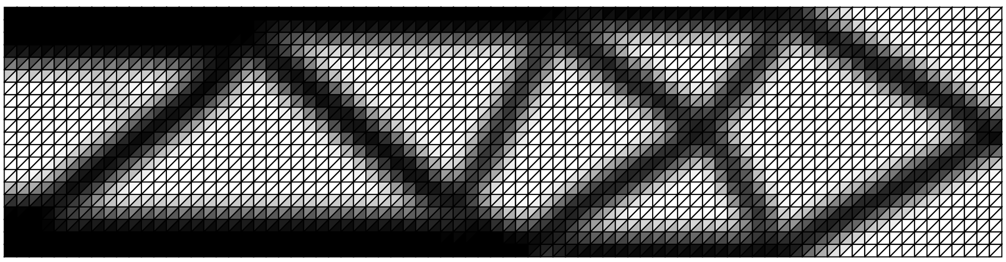

Figure 4.14: Plastic topology optimization with the presented formulation

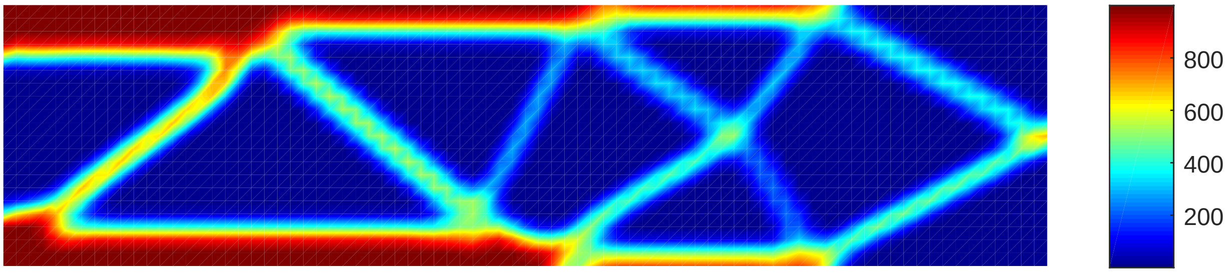


Figure 4.15: Stress map of the optimum topology of long cantilever beam problem

# 5
# Conclusions

It has become clear that topology optimization considering limit analysis is more advantageous than stress constrained topology optimization, since plastic design fully exploits material strength. Additionally, it is important to remark that stress constrained optimization, in general, relies on several parameters, due to penalization and aggregation. Those parameters influence deeply the final topology. On the other hand, topology optimization considering limit analysis is a fairly parameter-free method.

Furthermore, topology optimization considering limit analysis has also proved to be more suitable for practical applications than the standard topology optimization, since no prescribed volume constraint is required. Additionally, another advantage of this approach is that the design criterion is intrinsically verified.

It is worth mentioning that this formulation is easily extendable for three dimensional problems, since there are several papers dealing with limit analysis for three dimensional models. Another interesting future development of this field would be to address two materials, one being ductile and the other brittle.

In order to make possible to perform the optimization in more complex structures, further research is necessary in how to accelerate the solution. This might be achieved, for example, through adapting the solution of previous steps in order to find a warm start for the subproblem solution.

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