Comparing improved crack tip plastic zone estimates considering corrections based on T-stresses and on complete stress fields

J.T.P. Castro

Department of Mechanical Engineering, Pontifical Catholic University of Rio de Janeiro, PUC-Rio (Brazil)
jtcastro@puc-rio.br

R.A. Sousa

Department of Civil Engineering, Pontifical Catholic University of Rio de Janeiro, PUC-Rio (Brazil)
rflsousa@tecgraf.puc-rio.br

A.A.O. Lopes

Petrosoft Design, Rio de Janeiro (Brazil)
aalopes.petrosoft@petrobras.com.br

L.F. Martha

Department of Civil Engineering, Pontifical Catholic University of Rio de Janeiro, PUC-Rio (Brazil)
lfm@tecgraf.puc-rio.br

ABSTRACT. Ductile cracked structures yield locally under load to form a plastic zone (pz) around their crack tips. As the crack behavior strongly depends on this pz size, and as most cracked structures design routines depend on it, its precise estimation is a problem of major practical importance. The first classical pz estimates are based only on the stress intensity factor (SIF), but their precision is limited to very low nominal stresses. Improved estimates have been proposed considering the T-stress, obtained from the Williams series zero order term. However, neither SIF, nor SIF+T-stress based estimates can reproduce linear elastic (LE) stress fields that satisfy all boundary conditions in cracked components. In particular, the nominal stresses far from the crack tip, which have a major influence on the predicted pz size and shape. To prove this point, this paper first presents the complete LE stress field solution for the Griffith plate, using three different methods to arrive in the same analytical solution: the first is based on its Westergaard stress function, the second starts from the equivalent Inglis plate (considering its elliptical notch root as equal to about half the crack tip opening displacement), and the third is based on the complete Williams series. Next it introduces the equilibrium corrections necessary to compensate for the stress limitation inside the plastic zone. Then it compares pz estimates generated from this complete solution with classical SIF and SIF+T-stress pz estimates for various nominal stress to yield strength ratios, demonstrating the importance of using correct stress fields to evaluate such pz, particularly for the relatively high ratios used in high-performance structures. Finally, it speculates that for more complex structures, where the component geometry and type of loading may also significantly influence pz sizes and shapes, the plastic zones can be better estimated by a similar approach.

KEYWORDS. Complete Westergaard stress function; T-stress; Crack tip plastic zone.

INTRODUCTION

The SIF alone cannot model well some simple crack problems. E.g. the linear elastic (LE) stress field generated by a SIF $K_I = \sigma_n \sqrt{\pi a}$ in a Griffith plate with a *2a* crack, loaded in mode I by a nominal stress σ_n, does not obey the boundary conditions far from the crack tip: $\sigma_{ij} = [K_I / \sqrt{(2\pi r)}]g_{ij}(\theta) \Rightarrow \sigma_\theta(r \to \infty, 0) = 0$, instead of $\sigma_\theta(r \to \infty, 0) = \sigma_n$ as needed, where *r* is the distance from the tip, θ is the angle measured from the crack plane and $g_{ij}(\theta)$ are the Irwin θ-functions. LE analysis obviously cannot describe stresses and strains inside plastic zones *pz(θ)* around crack tips either. But both for teaching and designing purposes, *pz(θ)* are traditionally estimated from simplified LE analysis, assuming they depend only on K_I (in mode I). Indeed, equating the LE Mises stress to S_Y, the yielding strength, the simplest mode I elastic-plastic frontiers in plane stress (*pl-σ*) and in plane strain (*pl-ε*) are estimated by [1]

$$pz(\theta)_{pl-\sigma} = (K_I^2 / 2\pi S_Y^2) \cdot \cos^2(\theta/2) \cdot [1 + 3\sin^2(\theta/2)]$$
$$pz(\theta)_{pl-\varepsilon} = (K_I^2 / 2\pi S_Y^2) \cdot \cos^2(\theta/2) \cdot [(1 - 2v)^2 + 3\sin^2(\theta/2)]$$

(1)

where v is Poisson's coefficient. Thus, according to this classical estimate, the *pz(θ)* size directly ahead of crack tips in pl-σ, the reference used here to normalize *pz* plots, should be $pz(0)_{pl-\sigma} = pz_0 = (1/2\pi)(K_I/S_Y)^2$. But the $\sigma_{ij} = f(K_I)$ hypothesis is exact only when $r \to 0$, or exactly where the assumed LE behavior has no sense. Singular elastic-plastic (EP) estimates, such as the HRR field, do not solve this problem either. As the *pz* border may not be too close to crack tips, it is worth to at least estimate the effect of σ_n/S_Y on *pz(θ)*, where S_Y is the yielding strength, instead of simply neglecting it. A simplistic but clear estimate for this σ_n/S_Y effect can be made forcing $\sigma_y(x \to \infty, y = 0) = \sigma_n$, adding up a constant $\sigma_y = \sigma_n$ stress to the Williams (or Irwin) stress LE field to obtain

$$\sigma(\theta)_{M, pl-\sigma}^{Wil+\sigma_n} = [(\kappa g_x)^2 + (\kappa g_y + \sigma_n)^2 - (\kappa g_x)(\kappa g_y + \sigma_n) + 3(\kappa g_{xy})^2]^{1/2}$$

(2)

where $\sigma(\theta)_{M, pl-\sigma}^{Wil+\sigma_n}$ is the resulting LE Mises stress distribution around the crack tip in *pl-σ* (considering the σ_n/S_Y effect), $\kappa = K_I / \sqrt{(2\pi r)}$, and $g_x(\theta)$, $g_y(\theta)$, and $g_{xy}(\theta)$ are the mode I θ-functions associated with σ_x, σ_y and τ_{xy}. A similar equation can be easily generated for *pl-ε*. The corresponding *pz(θ)* are obtained from $\sigma_M(\theta) = S_Y$, see Fig. 1.
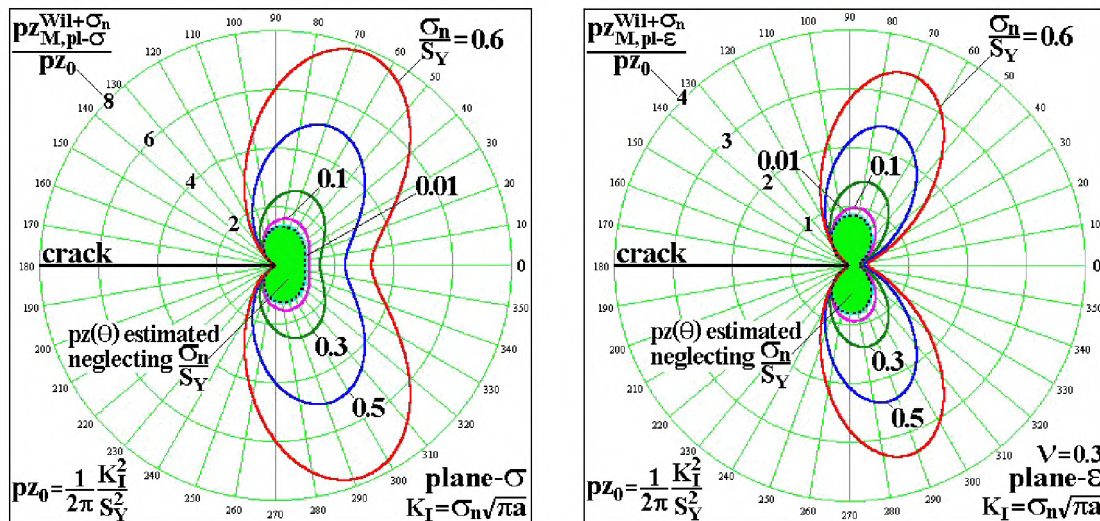

Figure 1: Mode I *pz(θ)* roughly estimated for the Griffith plate by $\sigma(\theta)_{M, pl-\sigma}^{Wil+\sigma_n} = S_Y$ for *pl-σ* and *pl-σ* limit conditions.

Fig. 1 indicates that the σ_n/S_Y ratio may significantly affect $pz(\theta)$ under real loading conditions, since engineering structures are typically designed with yield safety factors $1.2 < \phi_Y < 3$. However, it cannot prove that the σ_n/S_Y effects are that important, since the hypothesis used to generate this plots is not sound. But this simplistic estimate points out that the $pz(\theta)$ dependence on σ_n/S_Y should be further explored, as done in the following sections.

PLASTIC ZONES ESTIMATED USING THE INGLIS STRESSES

A much better estimate for the σ_n/S_Y effect on $pz(\theta)$ is obtained from the Inglis plate with a very sharp elliptical notch of major semi-axis a normal to σ_n, and minor semi-axis $b \ll a$. Making $x = c \cdot cosh\alpha \cdot cos\beta$ and $y = c \cdot sinh\alpha \cdot sin\beta$, this notch is described in elliptical coordinates (α,β) by $\alpha = \alpha_0$, where $a = c \cdot cosh\alpha_0$, $b = c \cdot sinh\alpha_0$, and $c = a/cos\alpha_0$. The general LE stress field in Inglis plates is given by a series too long to be reproduced here [2]. If the very sharp notch has a tiny (but finite) tip of radius $\rho = b^2/a = CTOD/2 = 2K_I^2/\pi S_Y E'$, where $E'=E$ in pl-σ and $E'=E/(1-v^2)$ in pl-ε, then its stress concentration factor $K_t = 1 + 2a/b$ is given by

$$K_t = 1 + 2 \cdot \frac{a}{b} = 1 + 2\sqrt{\frac{a}{\rho}} = 1 + 2 \cdot \sqrt{\frac{a\pi E'S_Y}{2 \cdot \sigma_n^2 \pi a}} \Rightarrow \frac{a}{b} = \sqrt{\frac{E'}{2 \cdot \sigma_n} \cdot \frac{S_Y}{\sigma_n}} = \sqrt{\frac{E'\phi_Y}{2 \cdot \sigma_n}} \tag{3}$$

Using this a/b ratio to obtain the notch shape that simulates the crack by $\alpha_0 = tanh^{-1}(b/a)$, then the LE stresses in the Inglis plate can be calculated. Finally, the Mises stress resulting from σ_α, σ_β, $\tau_{\alpha\beta}$, and $\sigma_z = v(\sigma_\alpha + \sigma_\beta)$ can be used to estimate the Inglis plastic zones by numerically solving equation (5) for $|\theta| \leq \pi$, see Fig. 2.

$$\begin{cases} \sigma_{M,pl-\sigma}^{Ing} = \sqrt{\sigma_\alpha^2 + \sigma_\beta^2 - \sigma_\alpha\sigma_\beta + 3\tau_{\alpha\beta}^2} = S_Y \\ \sigma_{M,pl-\varepsilon}^{Ing} = \sqrt{0.5[(\sigma_\alpha - \sigma_\beta)^2 + (\sigma_\alpha - \sigma_z)^2 + (\sigma_z - \sigma_\beta)^2] + 3\tau_{\alpha\beta}^2} = S_Y \end{cases} \tag{4}$$
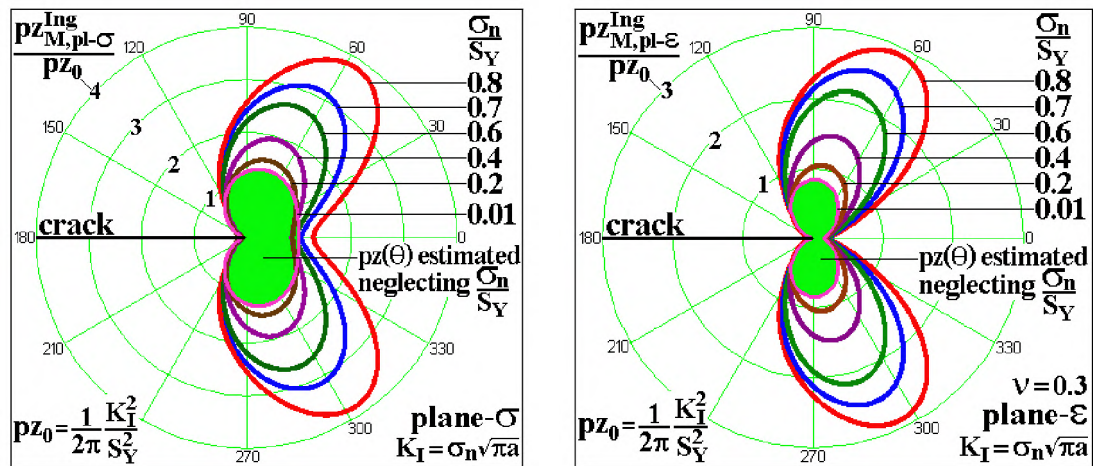

Figure 2: Mises plastic zones $pz(\theta)$ in pl-σ and pl-ε, calculated from the Inglis LE stress field for a cracked plate loaded in mode I, modeling the crack as a very sharp elliptical notch of tip radius $\rho = CTOD/2$.

Therefore, the influence of the nominal stress on Griffith's plate $pz(\theta)$, although a little less than estimated by the simplistic Fig.1 approximation, is indeed significant and should not be neglected in practical applications. Note that to use Inglis to obtain an exact LE stress field for the infinite cracked plate in mode I, when the crack is modeled as an elliptical sharp notch of tip radius $\rho = CTOD/2$, a quite reasonable hypothesis, since ideal cracks should open by CTOD under load. Nevertheless, it is worth to use an alternative approach to confirm it, as follows.

PLASTIC ZONES ESTIMATED USING THE WESTERGAARD STRESS FUNCTION

The Westergaard $Z(z)$ stress function provides an alternative way to rigorously estimate $pz(\theta)$ from the elastic stress field [3]. But, since the elastic-plastic frontier is not adjacent to the crack tip, the full stresses generated from $Z(z)$ must be used in such a calculation. This is easily demonstrated revisiting the classical Irwin solution for the Griffith plate loaded in mode I. Thus, if (x, y) and (r, θ) are Cartesian and polar coordinates centered at the crack tip, $i = \sqrt{-1}$ and $z = x + iy$ is a complex variable, the Irwin solution is obtained from the Westergaard stress function

$$Z(z) = z\sigma_n / \sqrt{(z^2 - a^2)} \Rightarrow Z'(z) = dZ/dz = -a^2\sigma_n / (z^2 - a^2)^{3/2} \tag{5}$$

$$\sigma_x = Re(Z) - y\,Im(Z') - \sigma_n, \quad \sigma_y = Re(Z) + y\,Im(Z'), \quad \tau_{xy} = -y\,Re(Z') \tag{6}$$

Note that to solve the mode I problem from $Z(z)$ a constant term $-\sigma_n$ has to be summed to $\sigma_x = Re(Z) - y\,Im(Z')$ to force $\sigma_x(\infty) = 0$ in the plate, an adequate mathematical trick since a constant stress in the x direction does not affect the stress field near the crack tip. However, the $\sigma_y = Re(Z) - y\,Im(Z')$ stress is usually approximated to generate a SIF (a highly desirable feature but for estimating $pz(\theta)$, since it neglects the σ_n / S_Y effect) by writing

$$\sigma_y(\theta = 0) = \sigma_n(x + a) / [(x + a)^2 - a^2]^{1/2} \cong \sigma_n a / \sqrt{(2ax)} = K_I / \sqrt{(2\pi r)} \quad (\text{if } x << a) \tag{7}$$

where $2a$ is the crack size perpendicular to σ_n. As (7) formally yields $\sigma_y(\theta = 0) = K_I / \sqrt{(2\pi r)} = 0$ if $r \rightarrow \infty$, this classical approximation obviously cannot be used to study the σ_n / S_Y influence on $pz(\theta)$. But this task can be fulfilled by first calculating the complete stress field generated from Z and Z' to obtain the resulting Mises (or Tresca, for that matter) stress, and then equating it to S_Y to obtain the required $pz(\theta)$ EP frontiers considering the σ_n / S_Y effect. The same process can be easily applied in pl-ε, see Fig.3. Inglis and Westergaard pz visually coincide when the sharp ellipse has its minor semi-axis (instead of its tip radius) $b = CTOD/2 = 2K_I^2/\pi S_Y E'$, see Fig. 4. As $pz_{Ing}(\theta)$ and $pz_{Wtg}(\theta)$ are obtained from completely different equations, their near coincidence is certainly not fortuitous. Therefore, the large σ_n / S_Y effect predicted by these rigorous solutions really should not be neglected in practice. This point must be emphasized for design purposes, since it is the plastic zone size that underlines most LEFM predictions.
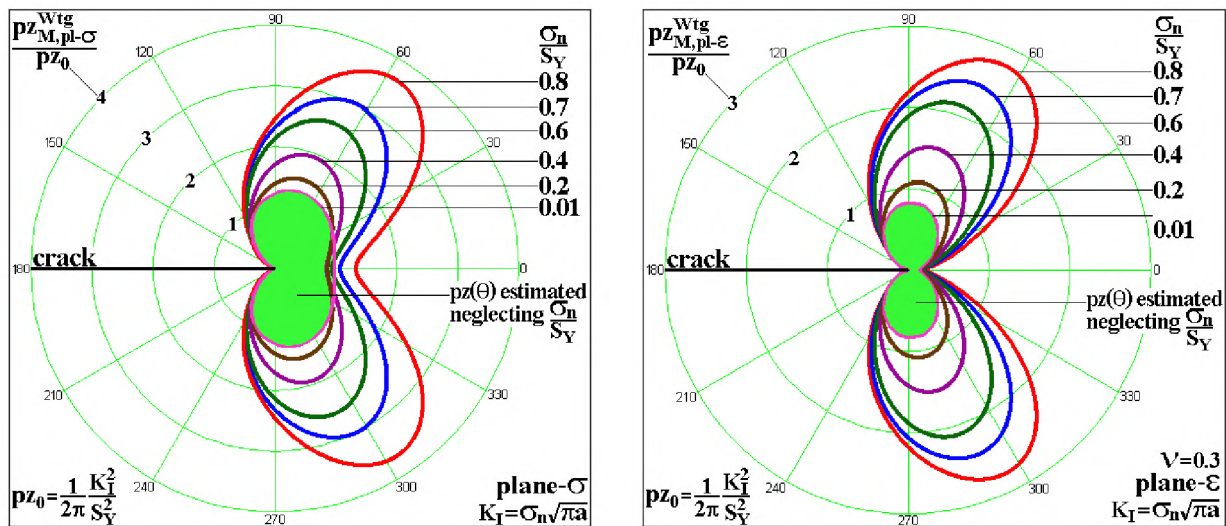

Figure 3: Mises $pz(\theta)$ for the Griffith plate loaded in mode I, estimated from the complete LE stress field induced by the Westergaard stress function for pl-σ and pl-ε conditions.

PLASTIC ZONES ESTIMATED USING THE COMPLETE WILLIAMS SERIES

The Williams series may be used to obtain exact LE stress fields, thus its coefficients may be adjusted to the complete field generated from the Westergaard stress function, successively incrementing its number of terms [4]. Fig. 5 shows the EP frontiers ahead of the crack tip obtained considering 1 to 4 terms. 3 terms are already sufficient to reproduce $pz_{Ing}(\theta) = pz_{Wtg}(\theta)$. Thus, exactly as expected, these three paths lead to the same $pz(\theta)$ estimations. These estimates are based on the Griffith plate correct LE stress field, which obeys the plate boundary conditions (namely $\sigma_y(|x| \leq a, y = 0) = \tau_{xy}(|x| \leq a, y = 0) = \sigma_x^\infty = \tau_{xy}^\infty = 0$, $\sigma_y^\infty = \sigma_n$). Thus they are the best $pz(\theta)$ LE estimate obtainable for the Griffith plate without considering equilibrium requirements. However, as the stresses inside the plastic zone are limited by yielding, the truncated LE stress field cannot obey equilibrium conditions. But such conditions can have a major influence on $pz(\theta)$, as recognized by Irwin a long time ago. The next topic considers them, and compares the resulting $pz(\theta)$ estimations with $pz(\theta)$ estimated considering only the T-stress correction.
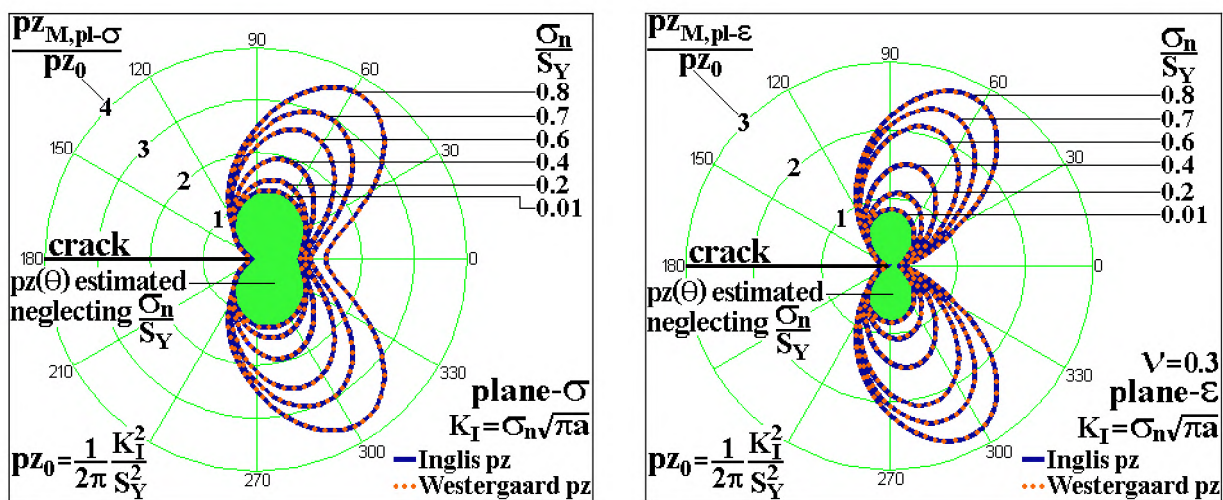

Figure 4: Mises $pz(\theta)$ estimated from the complete Westergaard stress field are visually identical to the Inglis estimate when a sharp elliptical notch with $b = CTOD/2 = 2K_I^2/\pi S_Y E'$ instead of $\rho = CTOD/2$ is used to model the crack.
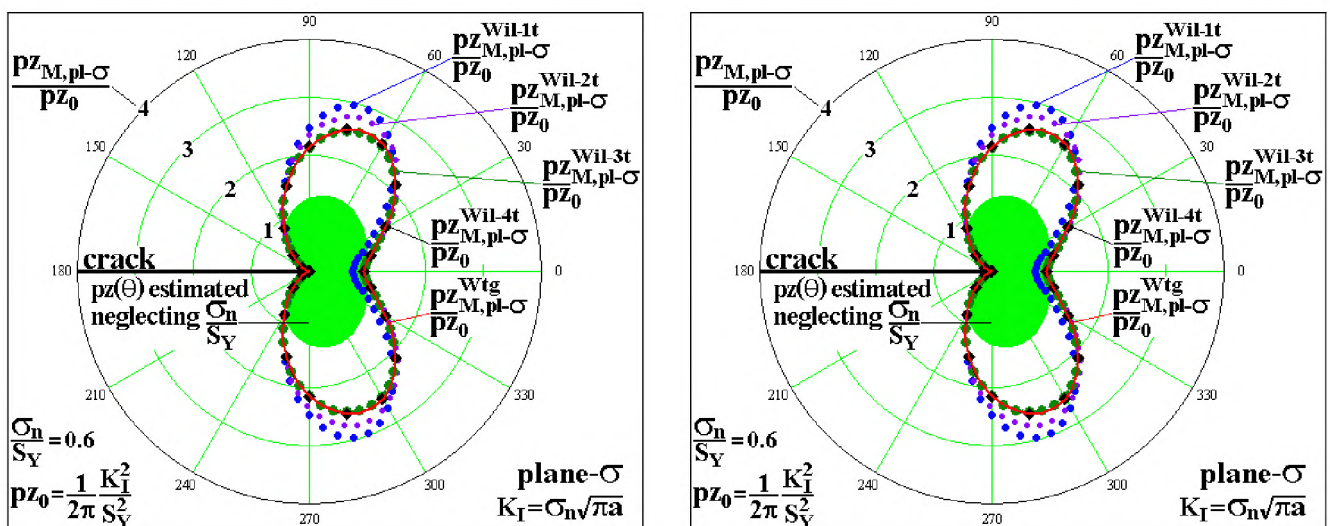

Figure 5: The Mises $pz(\theta)$ estimated for the Griffith plate loaded in mode I from the Williams series with only 3 terms visually reproduces reasonably well $pz_{Ing}(\theta) = pz_{Wtg}(\theta)$ both in pl-σ and in pl-ε.

T-STRESS AND EQUILIBRIUM INFLUENCE ON PLASTIC ZONE ESTIMATIONS

The T-stress correction is a constant σ_x term (parallel to the crack) added to the K_I-based stress field which can alleviate some of its limitations [5]. Thus, it has been widely explored in the literature to model some interesting problems [6-15]. From a practical point of view, Fett [16] lists T-stress values for several geometries. However, the resulting K_I+T-stress field cannot reproduce the $\sigma_y^\infty = \sigma_n$ boundary condition in the Griffith plate, as it is just a simplification of the complete stress field used above. Therefore it is interesting to compare the plastic zones estimated from these two LE fields.

But before doing so, it is important to remember that although the complete field generated e.g. from the Westergaard stress function is the correct LE solution for the Griffith plate, its truncation inside the plastic zone limits stresses, thus inevitably leads to underestimated *pz(θ)* frontiers. In a first approximation, such stresses can be limited by S_Y, neglecting strain-hardening effects inside *pz(θ)*, but such effects can be considered assuming an HRR-like stress-strain relation. However, due to space limitations only the ideal perfectly plastic behavior is discussed here.

Four alternative models for compensating the stress truncation inside the plastic zones augmenting them by forcing the plate to obey equilibrium conditions are considered following.

- Correction to compensate for the σ_y component truncation, as proposed by Rodriguez et al [17]:

$$pz_M^{Wtg+eq\sigma_y}(\theta) = \frac{\int_0^{pz_M^{Wtg}(\theta)} \{Re[\,Z(r,\theta)+y\,Im[\,Z'(r,\theta)\,]\}\ dr}{\sigma_y\left(pz_M^{Wtg}(\theta),\theta\right)} \tag{8}$$

This correction may be seen as a generalization of Irwin´s classical correction for the plastic zone along the crack direction $\theta = 0$, which is based on the equilibrium of net vertical forces that could not exist within the plastic zone because σ_y cannot surpass the yielding stress [3]. Besides the generalization to perform this correction along any θ-direction, the most important difference between equation (8) and Irwin´s receipt is that the former is based on the complete Westergaard stress function while the latter considers a stress field that is based solely on the SIF.

- Correction using a constant increment along each radius connecting the crack tip to the *pz(θ)* borderline, defined by its θ-direction, obtained from

$$pz_M^{Wtg+eqR}(\theta) = pz_M^{Wtg}(\theta) + CTE \tag{9}$$

where $CTE = pz_M^{Wtg+eqR}(\theta=0) - pz_M^{Wtg}(\theta=0)$. This constant has the same equilibrium rational as the previous correction, and it is based on the $\theta=0$ direction. For other radial directions, the same length correction is adopted, inspired by the idea of the constant T-stress correction.

- Correction based on the Mises stress, obtained from

$$pz_M^{Wtg+eqM}(\theta) = \frac{\int_0^{pz_M^{Wtg}(\theta)} \sigma_{Mises}(r,\theta)\ dr}{S_Y} \tag{10}$$

Since the correction $pz_M^{Wtg+eq\sigma_y}(\theta)$ only presents the equilibrium rational for $\theta=0$ and does not take into account the effect of the other stress components, this correction based on the Mises stress may be seen as a reasonable alternative, since it considers them and can be used for any type of loading.

- Correction based on the vertical traction component, obtained from

$$pz_M^{Wtg+eqTr}(\theta) = \frac{\int_0^{pz_M^{Wtg}(\theta)} t_y(r,\theta)\ dr}{t_y\left(pz_M^{Wtg}(\theta),\theta\right)} \tag{11}$$

where t_y is determined by

$$\begin{Bmatrix} t_x(r,\theta) \\ t_y(r,\theta) \end{Bmatrix} = \begin{bmatrix} \sigma_x(r,\theta) & \tau_{xy}(r,\theta) \\ \tau_{xy}(r,\theta) & \sigma_y(r,\theta) \end{bmatrix} \begin{Bmatrix} \cos(\theta) \\ \sin(\theta) \end{Bmatrix} \tag{12}$$

Again, this correction has an exact equilibrium appeal only for $\theta = 0$. However, by considering the vertical traction component, the equilibrium may be seen as resulting from a free body diagram obtained by sectioning the model along any θ-direction.

Fig. 6 and Fig. 7 compare the equilibrium various correction described above, by showing the difference between their $pz(\theta)$ estimates for plane stress and plane strain. These figures also depict plastic zones obtained by truncated SIF, SIF plus T-stress, and complete LE stress fields, which do not obey equilibrium requirements. Note in particular that the K_I+T-stress $pz(\theta)$ is significantly smaller than equilibrium-corrected ones.
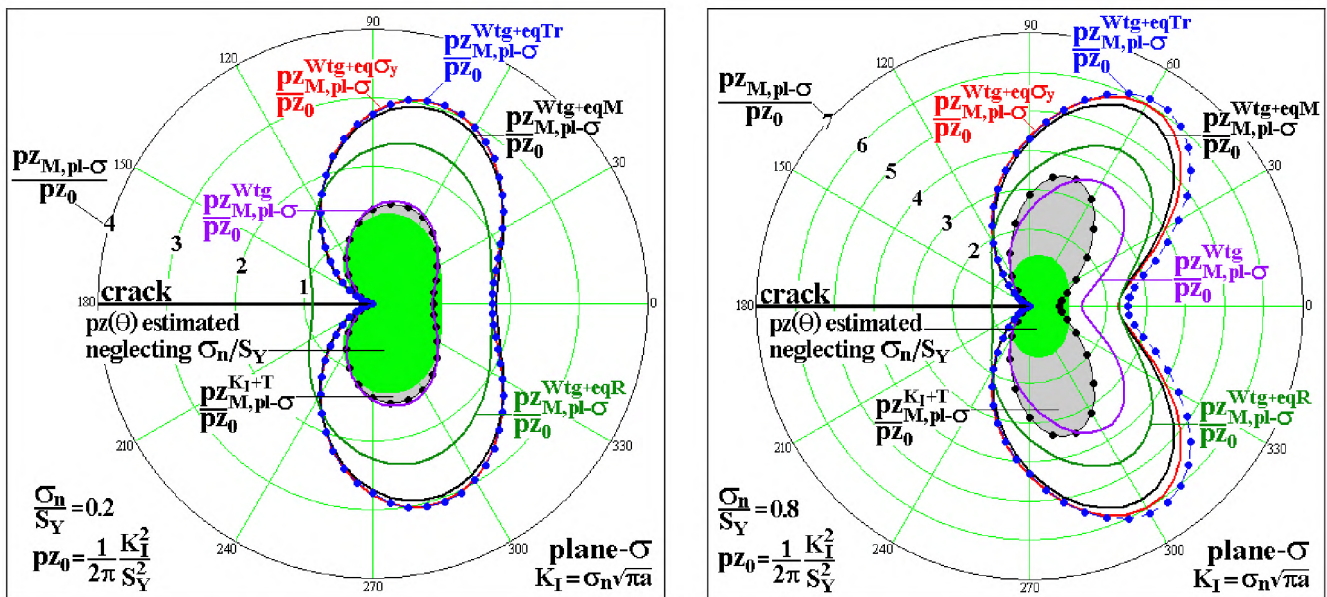

Figure 6: Equilibrium-corrected $pz(\theta)$ and $pz^{KI+T}(\theta)$ estimated for the Griffith plate loaded in mode I in *pl-σ*.
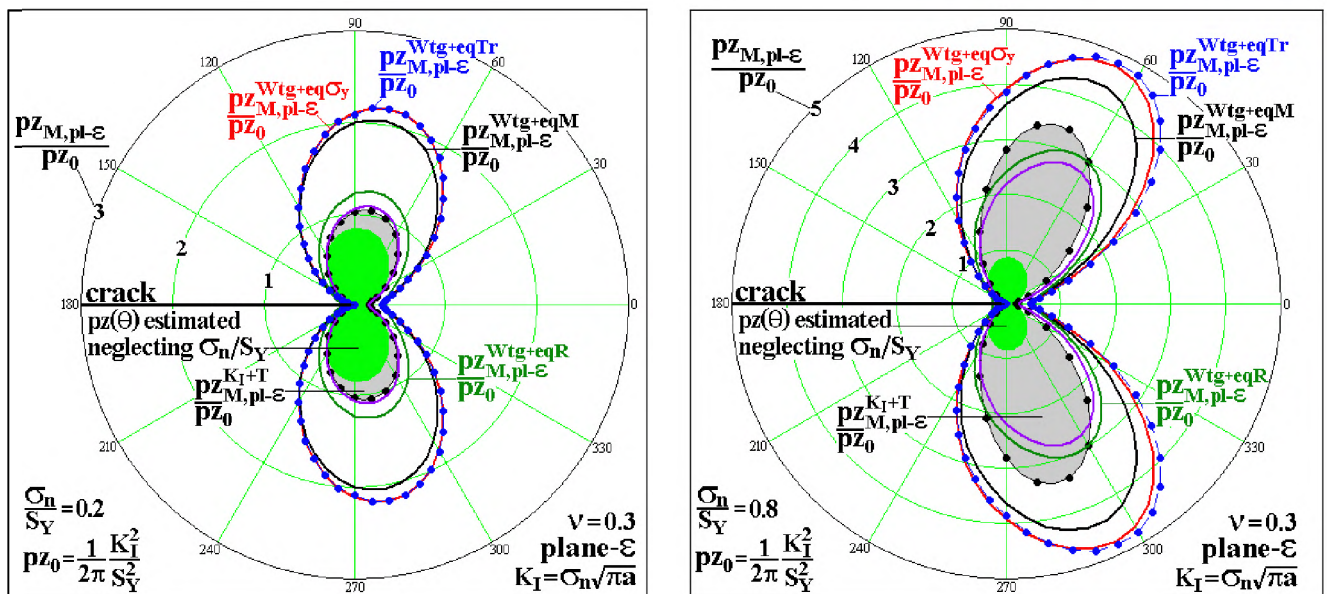

Figure 7: Equilibrium-corrected $pz(\theta)$ and $pz^{KI+T}(\theta)$ estimated for the Griffith plate loaded in mode I in *pl-ε*.

Fig.6 and Fig.7 display $p\chi(\theta)$ borderlines estimated for $\sigma_n/S_Y = 0.2$ and 0.8 ratios, which correspond to yield safety factors $\phi_Y = 5$ and 1.25, representative of maxima low and high loads used in typical structural applications. The equilibrium-corrected hypothesis based on σ_j, σ_M, and on the traction vector provide similar $p\chi(\theta)$ predictions, which are significantly larger than the K_I+T-stress one usually accepted as reliable $p\chi(\theta)$ estimates for analysis and design purposes. As the K_I+T-stress field neglects stress components considered by the exact LE solution for the Griffith plate, this suggest that for practical applications $p\chi(\theta)$ in generic cracked components should be estimated using equilibrium-corrected LE stress fields properly calculated using standard finite element procedures [18].

CONCLUSIONS

The nominal stress to yield strength ratio significantly affects the size and shape of plastic zones ahead of crack tips estimated from LE stress fields, as demonstrated for Griffith's plate using 3 different ways to find its exact solution. This solution should be corrected to consider equilibrium requirements violated by the LE stress truncation inside the plastic zone, a task tackled by 4 different approximate but reasonable hypotheses. From these, the stress-based ones generate quite similar $p\chi(\theta)$ estimates. Such equilibrium-corrected $p\chi(\theta)$ are significantly larger than the estimates obtained from the plate SIF $K_I = \sigma_n\sqrt{(\pi a)}$ alone, or from the combination of its SIF+T-stress, particularly for the high σ_y/S_Y ratios used in modern structures. As such estimates are based on an exact LE solution complemented by sensible equilibrium assumptions, they indicate that the traditional practice of assuming that T-stress can adequately correct SIF limitation for estimating $p\chi(\theta)$ may, and probably should be questioned. Moreover, they suggest that $p\chi(\theta)$ frontiers can be similarly estimated in cracked structural components using complete LE stress fields calculated by well-established finite element procedures, which should be then equilibrium-corrected to avoid underestimation due to stress truncation, possibly including strain-hardening effects for better precision. This fact has important practical consequences, as it can be used to seriously question the similitude principle, one milestone of the mechanical design against fracture, in many real life problems. In compensation, it may help to better predict the actual toughness of real structures, by comparing reliable estimates for their $p\chi(\theta)$ with those obtained for the standard test specimens used to measure their material toughness.

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