2D Computer Graphics

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IMPA

Path representation

SVG path commands

Command		Parameters	Description
Abs	Rel		
`M`	`m`	$(x, y)+$	move
`L`	`l`	$(x, y)+$	line
`H`	`h`	$x+$	horizontal line
`V`	`v`	$y+$	vertical line
`C`	`c`	$(x_1, y_1, x_2, y_2, x, y)+$	cubic
`S`	`s`	$(x_2, y_2, x, y)+$	smooth cubic
`Q`	`q`	$(x_1, y_1, x, y)+$	quadratic
`T`	`t`	$(x, y)+$	smooth quadratic
`A`	`a`	$(r_x, r_y, \theta_x, \ell, o, x, y)+$	elliptical arc
`Z`	`z`		close path

Our representation

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path_data = shape:as_path_data()
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Content visible using *iterators*

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path_data = shape:as_path_data()
```

Content visible using *iterators*

```
path_data:iterate{
  begin_contour = function(self, x0, y0) end
  end_open_contour = function(self, x0, y0) end
  end_closed_contour = function(self, x0, y0) end
  linear_segment = function(self, x0, y0, x1, y1) end
  quadratic_segment = function(self, x0, y0, x1, y1, x2, y2) end
  rational_quadratic_segment = function(self, x0, y0, x1, y1, w1, x2, y2) end
  cubic_segment = function(self, x0, y0, x1, y1, x2, y2, x3, y3) end
}
```

Example of filter

Transform a path and forward results on

```
function filter.make_input_path_f_xform(xf, forward)
  local px, py -- previous cursor
  local xformer = {}
  function xformer:begin_contour(x0, y0)
      px, py = xf:apply(x0, y0)
      forward:begin_contour(px, py)
  end
  function xformer:end_closed_contour(x0, y0)
      forward:end_closed_contour(px, py)
  end
  function xformer:linear_segment(x0, y0, x1, y1)
     x1, y1 = xf:apply(x1, y1)
     forward:linear_segment(px, py, x1, y1)
     px, py = x1, y1
  end
  function xformer:rational_quadratic_segment(x0, y0, x1, y1, w1, x2, y2)
      x1, y1, w1 = xf:apply(x1, y1, w1)
      x2, y2 = xf:apply(x2, y2)
      forward:rational_quadratic_segment(px, py, x1, y1, w1, x2, y2)
      px, py = x2, y2
  end
  ...
  return xformer
end
```

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Implement `accelerate` to convert and store in your representation

Chain transformation, monotonization, and acceleration

```
path_xf = shape:get_xf():transform(cur_xf)
shape:as_path_data():iterate(
  filter.make_input_path_f_xform(path_xf,
    monotonize(
      accelerate(accel)))
```

Floating-point and root-finding

All real numbers represented in a computer?

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$$N_A = 6.022140857 \times 10^{23} \qquad q_e = 1.60217662 \times 10^{-19}$$

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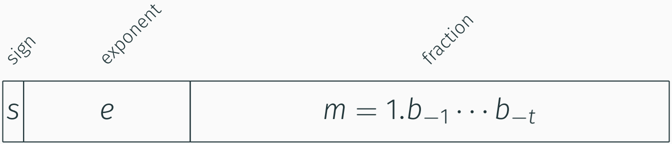
Except, in binary...

Binary floating-point

$$(-1)^s \times (1.b_{-1}b_{-2}b_{-3}\cdots b_{-t})_2 \times 2^{e-z} \qquad z = 2^{w-1} - 1$$



One *sign* bit

w exponent bits

t fraction bits

$$(-1)^s \times (1.b_{-1}b_{-2}b_{-3}\cdots b_{-t})_2 \times 2^{e-z} \qquad z = 2^{w-1} - 1$$

Normalized representation for mantissa m

- Ensures unique representation for mantissa

$$1_2 \leq m < 10_2$$

- First bit is implicitly set to 1

Representation details

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Excess encoding for exponent e

- Allows for positive and negative exponents
- Therefore large and small magnitudes
- Subtract $z = 2^{w-1} - 1$ from encoded exponent

Special values

Largest representable exponent is reserved

- $m = 0$ represents $\pm Inf$ (Infinity)
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Other special operations

$$x \div Inf = \pm 0 \qquad 0 \div 0 = NaN$$

$$Inf \times Inf = Inf \qquad Inf - Inf = NaN$$

$$x \div 0 = \pm Inf \qquad Inf \div Inf = NaN$$

$$Inf + Inf = Inf \qquad Inf \times 0 = NaN$$

Denormalization

$$(-1)^s \times (1.b_{-1}b_{-2}b_{-3}\cdots b_{-t})_2 \times 2^{e-z} \qquad z = 2^{w-1} - 1$$

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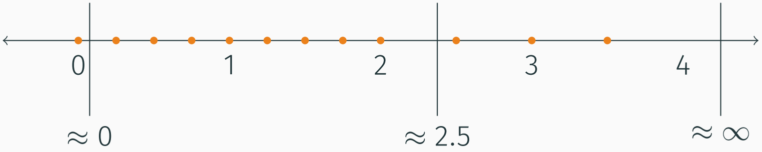
Summary of representation

	sign	exponent	fraction
normalized	±	not $0\cdots0$ or $1\cdots1$	any
de-normalized	±	$0\cdots0$	not $0\cdots0$
zero	±	$0\cdots0$	$0\cdots0$
Inf	±	$1\cdots1$	$0\cdots0$
NaN	±	$1\cdots1$	not $0\cdots0$

	Single precision	Double precision
Total bits	32	64
Exponent bits	8	11
Fraction bits	23	52
Exponent range	-126...127	-1022...1023
Smallest magnitude	$\approx 10^{-45}$	$\approx 10^{-324}$
Decimal range	$\approx [-10^{38}, 10^{38}]$	$\approx [-10^{308}, 10^{308}]$
Decimal precision	7	16

Rounding, overflow, underflow



Let's try to represent 0.1 in floating-point

- Fraction is 0.00011001100110011001100...
- No exact representation possible

Errors can grow and dominate results

Problem often happens in practice

Source of arithmetic errors

Addition may not be exact even when exponents are equal

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- Must *pre-shift* to match exponents
- $1.0000 + 1.0000 \times 2^{-5} = 1.00001 \to 1.0000$
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Multiplication is even worse

- Even with matching exponents, needs double number of bits!

Other weirdness

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Equality operator is basically useless

- Returns true only when *exactly* equal
- Must use special function

The only guarantee is the following

$$fl(x \odot y) = (x \odot y)(1 + \delta_1), \qquad |\delta_1| \leq u = 2^{-t}$$

$$fl(x \odot y) = \frac{x \odot y}{1 + \delta_2}, \qquad |\delta_2| \leq u$$

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Safe Newton-Raphson

- Combines advantages of both methods

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What about in Bernstein basis?

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