# 2D Computer Graphics

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IMPA

# Geometry and Transformations

# Cartesian coordinate system

Points defined by pair of coordinates

- Signed distances to perpendicular directed lines
- Point where lines cross is the origin

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- Connection between Euclidean geometry and algebra

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- Signed distances to perpendicular directed lines
- Point where lines cross is the origin

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- Connection between Euclidean geometry and algebra
- Describe shapes with equations
- E.g., lines and circles

Distance between line and point?

# Problems

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Find intersection between line and circle?

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Prove that the medians of a triangle are concurrent?

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Set of $V$ of *vectors* closed by linear combinations

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- That spans $V$

$$v \in V \;\Leftrightarrow\; \exists\, \alpha_1, \alpha_2 \;|\; v = \alpha_1 v_1 + \alpha_2 v_2$$

Coordinates of a vector in a given basis

$$[v]_{\mathcal{B}} = \begin{bmatrix} \alpha_1 \\ \alpha_2 \end{bmatrix} \Leftrightarrow v = \alpha_1 v_1 + \alpha_2 v_2$$

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Interesting transformations

- Identity, Rotation, Scale, Reflection, Shearing

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General linear group

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General linear group

- Composition, inverse
- Preserves collinearity, parallelism, concurrency, tangency, ratios of distances along lines

Dot product, scalar product, standard inner product

$$u^T v = u \cdot v = \langle u, v \rangle = u_x v_x + u_y v_y$$

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Let $u$ and $v$ make angles $\alpha$ and $\beta$ with the x-axis

$$\begin{aligned}\cos(\beta - \alpha) &= \cos(\alpha)\cos(\beta) + \sin(\alpha)\sin(\beta) \\ &= u_x/\|u\| v_x/\|v\| + u_y/\|u\| v_y/\|v\| \\ &= \langle u, v \rangle / (\|u\| \|v\|)\end{aligned}$$

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Affine space is $A = o + V = \{p \mid p - o \in V\}$

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- Points $p \in A$ are such that $\sum_{i=0}^{2} \alpha_i = 1$ (affine combination)

Combination of two arbitrary parallel projections

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Preserve affine combinations

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# Affine transformations

Interesting transformations

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What about the matrix in barycentric frame $\mathcal{D} = \{a_0, a_1, a_2\}$

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Visualization of the affine plane

Line $ax + by + c = 0$

$$n^T p = 0, \quad \text{with}$$

$$n^T = \begin{bmatrix} a & b & c \end{bmatrix} \quad \text{and} \quad p = \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

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# Revisiting problems

Distance between line and point?

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## $\mathbb{RP}^2$

Projective points: lines through origin in 3D

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Homogeneous coordinates

- Generalization of affine coordinates

$$\begin{bmatrix} a \\ b \\ c \end{bmatrix}, \quad a, b, c \text{ not all zero} \qquad \begin{bmatrix} w\,x \\ w\,y \\ w \end{bmatrix} \equiv \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}, \quad w \neq 0$$

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Matrix of a projective transformation

$$[T] = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

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Must be invertible

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# References

D. A. Brannan, M. F. Esplen, and J. J. Gray. *Geometry*. Cambridge University Press, 2011.