2D Computer Graphics

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Summer 2020

IMPA

Inflection points and double points

Covariant and contravariant tensors

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Line-like (plane-like) things are *covariant tensors*

Einstein's notation

Coordinates of contravariant tensors use superscripts $P = \begin{bmatrix} P^1 & P^2 & P^3 \end{bmatrix}$

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Whenever there is an expression with the same index name appearing as a subscript and a subscript, the summation sign is omitted

$$P^i L_i = P^1 L_1 + P^2 L_2 + P^3 L_3 = L_1 P^1 + L_2 P^2 + L_3 P^3 = L_i P^i$$

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The covariant index transforms with T and the contravariant index transforms with T^*

$$N_k^\ell = M_i^j T_k^i (T^*)_j^\ell$$

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That's why conics are weird! Both covariant indices transform with T

$$U_{k\ell} = Q_{ij}T^i_kT^j_\ell$$

The polar line

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Its coordinates are simply $L_j = Q_{ij}P^i$

Proof

$$Q_{ij}R^iR^j = 0 \quad \text{and} \quad Q_{ij}S^iS^j = 0$$

$$(Q_{ij}P^i)R^j = 0 \Leftrightarrow (Q_{ij}R^j)P^i = 0$$

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Can you interpret the point $P^j = (Q^*)^{ij} L_i$?

Generalized cross product

Is a function $\mathbf{cr}_n : (R^n)^{n-1} \to R_n$

$$L = \mathbf{cr}_n(\overbrace{P, Q, \ldots, R}^{n-1}), \quad \text{with} \quad L_i P^i = L_i Q^i = \cdots L_i R^i = 0$$

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Represents the “plane” that goes through all points, or the point of intersection of all planes

Trick is to use look at the determinants

$$\begin{vmatrix} P & P & Q & \cdots & R \end{vmatrix} = \begin{vmatrix} Q & P & Q & \cdots & R \end{vmatrix} = \cdots = \begin{vmatrix} R & P & Q & \cdots & R \end{vmatrix} = 0$$

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The expansion shows, as required, that

$$L_i P^i = L_i Q^i = \cdots L_i R^i = 0.$$

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$$\varepsilon_{\pi(1)\pi(2)\dots\pi(n)} = \sigma(\pi)$$

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Can be used to compactly represent the determinant

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$$\big(\mathrm{cr}_2(P)\big)_j = P^i \varepsilon_{ij} \quad \big(\mathrm{cr}_3(P,Q)\big)_k = P^i Q^j \varepsilon_{ijk} \quad \big(\mathrm{cr}_4(P,Q,R)\big)_\ell = P^i Q^j R^k \varepsilon_{ijk\ell}$$

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If you contract with any of the arguments, you get a determinant of a matrix with repeated column

Epsilon-delta rule

Useful relationship between Levi-Civita epsilon and Kronecker delta

$$\varepsilon_{k_1 k_2 \ldots k_n} \varepsilon^{\ell_1 \ell_2 \ldots \ell_n} = \det\left([\delta^{\ell_j}_{k_i}]_{ij}\right)$$

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If any indices repeat on either epsilon, you have zero on the left. On the right, you have either a repeated column or a repeated row. Either way, the determinant is also zero.

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If no indices repeat on either epsilon, you have the product of the signs of the permutations on the left. On the right, the matrix is an identity matrix with rows and columns permuted in the same way. The determinant is also the product of the signs of the permutations.

Epsilon-delta rule

A couple special cases

$$\varepsilon_{ij}\varepsilon^{i\ell} = \delta_j^\ell \qquad \varepsilon_{ijk}\varepsilon^{imn} = \delta_j^m\delta_k^n - \delta_j^n\delta_k^m$$

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Proof

$$\begin{aligned}
A_m(B^jC^k\varepsilon_{jki})\varepsilon^{min} &= A_mB^jC^k(-\varepsilon_{ijk}\varepsilon^{imn}) \\
&= A_mB^jC^k(\delta_j^n\delta_k^m - \delta_j^m\delta_k^n) \\
&= A_mB^jC^k\delta_j^n\delta_k^m - A_mB^jC^k\delta_j^m\delta_k^n \\
&= (A_mC^k\delta_k^m)(B^j\delta_j^n) - (A_mB^j\delta_j^m)(C^k\delta_k^n) \\
&= (A_mC^m)B^n - (A_mB^m)C^n
\end{aligned}$$

Also from epsilon-delta rule

$$\begin{aligned}(A \times B) \cdot (C \times D) &= (A \cdot C)(B \cdot D) - (A \cdot D)(B \cdot C) \\ &= \det\left(\begin{bmatrix} A \\ B \end{bmatrix} \begin{bmatrix} C & D \end{bmatrix}\right)\end{aligned}$$

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Proof

$$
\begin{aligned}
A_i B_j \varepsilon^{jik} C^m D^n \varepsilon_{mnk} &= A_i B_j C^m D^n (\varepsilon^{kji} \varepsilon_{kmn}) \\
&= A_i B_j C^m D^n (\delta^j_n \delta^i_m - \delta^j_m \delta^i_n) \\
&= A_i B_j C^m D^n \delta^j_n \delta^i_m - A_i B_j C^m D^n \delta^j_m \delta^i_n \\
&= (A_m C^m)(B_n D^n) - (A_n D^n)(B_m C^m)
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General case is also true! (We will use this shortly)

Let γ be a rational curve

$$\gamma(t) = \begin{bmatrix} x(t) & y(t) \end{bmatrix}^T$$

$$x(t) = \frac{u(t)}{w(t)} \qquad y(t) = \frac{v(t)}{w(t)}$$

Inflection points

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When curvature changes sign, i.e., speed and acceleration are collinear

$$\gamma'(t) \times \gamma''(t) = 0$$

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Same as condition

$$\begin{vmatrix} \alpha(t) & \alpha'(t) & \alpha''(t) \end{vmatrix} = 0, \quad \text{with}$$

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So “inflection” only when control points are linearly dependent

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So “inflection” only when control points are linearly dependent

The quadratic degenerates to a line, half a line, or a point

Not really an inflection

Cubics

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Maybe we should give up because the expression is unwieldy...

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$$\begin{vmatrix}\alpha(t) & \alpha'(t) & \alpha''(t)\end{vmatrix} = \left|\begin{bmatrix}p_0 & p_1 & p_2 & p_3\end{bmatrix}\begin{bmatrix}B_3(t) & B_3'(t) & B_3''(t)\end{bmatrix}\right|$$

Maybe we should give up because the expression is unwieldy...

Or we use Lagrange's identity to make it treatable

$$\begin{vmatrix}\alpha(t) & \alpha'(t) & \alpha''(t)\end{vmatrix} = \mathrm{cr}_4(x_{0-3}, y_{0-3}, w_{0-3}) \cdot \mathrm{cr}_4\big(B_3(t), B_3'(t), B_3''(t)\big)$$

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This the *inflection polynomial*—a cubic!

Inflections happen when t is a root

In Mathematica...

Show the inflection polynomial

A cubic that reduces to a quadratic in the integral case

Show that inflection points are collinear

Double-points in Mathematica

Given t_1 t_2 of double-point, then $\gamma(t_1), \gamma(t_2), \gamma(t_3)$ are collinear for all t_3

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Results in the double-point polynomial: a quadratic

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Results in the double-point polynomial: a quadratic

Compare the discriminants of the inflection polynomial and the double-point polynomial and use them to classify the cubics

References

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